

Credibility theory features of **actuar**

Christophe Dutang
ISFA, Université Claude Bernard Lyon 1

Vincent Goulet
École d'actuariat, Université Laval

Xavier Milhaud
ISFA, Université Claude Bernard Lyon 1

Mathieu Pigeon
École d'actuariat, Université Laval

1 Introduction

Credibility models are actuarial tools to distribute premiums fairly among a heterogeneous group of policyholders (henceforth called *entities*). More generally, they can be seen as prediction methods applicable in any setting where repeated measures are made for subjects with different risk levels.

The credibility theory features of **actuar** consist of matrix `hachemeister` containing the famous data set of [Hachemeister \(1975\)](#) and function `cm` to fit hierarchical (including Bühlmann, Bühlmann-Straub) and regression credibility models. Furthermore, function `simul` can simulate portfolios of data satisfying the assumptions of the aforementioned credibility models; see the "simulation" vignette for details.

2 Hachemeister data set

The data set of [Hachemeister \(1975\)](#) consists of private passenger bodily injury insurance average claim amounts, and the corresponding number of claims, for five U.S. states over 12 quarters between July 1970 and June 1973. The data set is included in the package in the form of a matrix with 5 rows and 25 columns. The first column contains a state index, columns 2–13 contain the claim averages and columns 14–25 contain the claim numbers:

```

> data(hachemeister)
> hachemeister

      state ratio.1 ratio.2 ratio.3 ratio.4 ratio.5 ratio.6 ratio.7
[1,]     1   1738   1642   1794   2051   2079   2234   2032
[2,]     2   1364   1408   1597   1444   1342   1675   1470
[3,]     3   1759   1685   1479   1763   1674   2103   1502
[4,]     4   1223   1146   1010   1257   1426   1532   1953
[5,]     5   1456   1499   1609   1741   1482   1572   1606
      ratio.8 ratio.9 ratio.10 ratio.11 ratio.12 weight.1 weight.2
[1,]    2035    2115    2262    2267    2517    7861    9251
[2,]    1448    1464    1831    1612    1471    1622    1742
[3,]    1622    1828    2155    2233    2059    1147    1357
[4,]    1123    1343    1243    1762    1306     407     396
[5,]    1735    1607    1573    1613    1690    2902    3172
      weight.3 weight.4 weight.5 weight.6 weight.7 weight.8 weight.9
[1,]    8706    8575    7917    8263    9456    8003    7365
[2,]    1523    1515    1622    1602    1964    1515    1527
[3,]    1329    1204     998    1077    1277    1218     896
[4,]     348     341     315     328     352     331     287
[5,]    3046    3068    2693    2910    3275    2697    2663
      weight.10 weight.11 weight.12
[1,]     7832     7849     9077
[2,]     1748     1654     1861
[3,]     1003     1108     1121
[4,]      384      321      342
[5,]     3017     3242     3425

```

3 Hierarchical credibility model

The linear model fitting function of R is named `lm`. Since credibility models are very close in many respects to linear models, and since the credibility model fitting function of **actuar** borrows much of its interface from `lm`, we named the credibility function `cm`.

Function `cm` acts as a unified interface for all credibility models supported by the package. Currently, these are the unidimensional models of [Bühlmann \(1969\)](#) and [Bühlmann and Straub \(1970\)](#), the hierarchical model of [Jewell \(1975\)](#) (of which the first two are special cases) and the regression model of [Hachemeister \(1975\)](#), optionally with the intercept at the barycenter of time ([Bühlmann and Gisler, 2005](#), Section 8.4). The modular design of `cm` makes it easy to add new models if desired.

This subsection concentrates on usage of `cm` for hierarchical models.

There are some variations in the formulas of the hierarchical model in the literature. We compute the credibility premiums as given in [Bühlmann and Jewell \(1987\)](#) or [Bühlmann and Gisler \(2005\)](#). We support three types of estimators of the between variance structure parameters: the unbiased estimators of [Bühlmann and Gisler \(2005\)](#) (the default), the slightly different version of

Ohlsson (2005) and the iterative pseudo-estimators as found in Goovaerts and Hoogstad (1987) or Goulet (1998).

Consider an insurance portfolio where contracts are classified into cohorts. In our terminology, this is a two-level hierarchical classification structure. The observations are claim amounts S_{ijt} , where index $i = 1, \dots, I$ identifies the cohort, index $j = 1, \dots, J_i$ identifies the contract within the cohort and index $t = 1, \dots, n_{ij}$ identifies the period (usually a year). To each data point corresponds a weight — or volume — w_{ijt} . Then, the best linear prediction for the next period outcome of a contract based on ratios $X_{ijt} = S_{ijt}/w_{ijt}$ is

$$\begin{aligned}\hat{\pi}_{ij} &= z_{ij}X_{ijw} + (1 - z_{ij})\hat{\pi}_i \\ \hat{\pi}_i &= z_iX_{izw} + (1 - z_i)m\end{aligned}\tag{1}$$

with the credibility factors

$$\begin{aligned}z_{ij} &= \frac{w_{ij\Sigma}}{w_{ijk\Sigma} + s^2/a}, & w_{ij\Sigma} &= \sum_{t=1}^{n_{ij}} w_{ijt} \\ z_i &= \frac{z_{i\Sigma}}{z_{i\Sigma} + a/b}, & z_{i\Sigma} &= \sum_{j=1}^{J_i} z_{ij}\end{aligned}$$

and the weighted averages

$$\begin{aligned}X_{ijw} &= \sum_{t=1}^{n_{ij}} \frac{w_{ijt}}{w_{ij\Sigma}} X_{ijt} \\ X_{izw} &= \sum_{j=1}^{J_i} \frac{z_{ij}}{z_{i\Sigma}} X_{ijw}.\end{aligned}$$

The estimator of s^2 is

$$\hat{s}^2 = \frac{1}{\sum_{i=1}^I \sum_{j=1}^{J_i} (n_{ij} - 1)} \sum_{i=1}^I \sum_{j=1}^{J_i} \sum_{t=1}^{n_{ij}} w_{ijt} (X_{ijt} - X_{ijw})^2.\tag{2}$$

The three types of estimators for parameters a and b are the following. First, let

$$\begin{aligned}A_i &= \sum_{j=1}^{J_i} w_{ij\Sigma} (X_{ijw} - X_{iww})^2 - (J_i - 1)s^2 & c_i &= w_{i\Sigma\Sigma} - \sum_{j=1}^{J_i} \frac{w_{ij\Sigma}^2}{w_{i\Sigma\Sigma}} \\ B &= \sum_{i=1}^I z_{i\Sigma} (X_{izw} - \bar{X}_{zzw})^2 - (I - 1)a & d &= z_{\Sigma\Sigma} - \sum_{i=1}^I \frac{z_{i\Sigma}^2}{z_{\Sigma\Sigma}},\end{aligned}$$

with

$$\bar{X}_{zzw} = \sum_{i=1}^I \frac{z_{i\Sigma}}{z_{\Sigma\Sigma}} X_{izw}.\tag{3}$$

(Hence, $E[A_i] = c_i a$ and $E[B] = db$.) Then, the Bühlmann–Gisler estimators are

$$\hat{a} = \frac{1}{I} \sum_{i=1}^I \max\left(\frac{A_i}{c_i}, 0\right) \quad (4)$$

$$\hat{b} = \max\left(\frac{B}{d}, 0\right), \quad (5)$$

the Ohlsson estimators are

$$\hat{a}' = \frac{\sum_{i=1}^I A_i}{\sum_{i=1}^I c_i} \quad (6)$$

$$\hat{b}' = \frac{B}{d} \quad (7)$$

and the iterative (pseudo-)estimators are

$$\tilde{a} = \frac{1}{\sum_{i=1}^I (J_i - 1)} \sum_{i=1}^I \sum_{j=1}^{J_i} z_{ij} (X_{ijw} - X_{izw})^2 \quad (8)$$

$$\tilde{b} = \frac{1}{I-1} \sum_{i=1}^I z_i (X_{izw} - X_{zzw})^2, \quad (9)$$

where

$$X_{zzw} = \sum_{i=1}^I \frac{z_i}{z_\Sigma} X_{izw}. \quad (10)$$

Note the difference between the two weighted averages (3) and (10). See [Belhadj et al. \(2009\)](#) for further discussion on this topic.

Finally, the estimator of the collective mean m is $\hat{m} = X_{zzw}$.

The credibility modeling function `cm` assumes that data is available in the format most practical applications would use, namely a rectangular array (matrix or data frame) with entity observations in the rows and with one or more classification index columns (numeric or character). One will recognize the output format of `simul` and its summary methods.

Then, function `cm` works much the same as `lm`. It takes in argument: a formula of the form `~` terms describing the hierarchical interactions in a data set; the data set containing the variables referenced in the formula; the names of the columns where the ratios and the weights are to be found in the data set. The latter should contain at least two nodes in each level and more than one period of experience for at least one entity. Missing values are represented by NAs. There can be entities with no experience (complete lines of NAs).

In order to give an easily reproducible example, we group states 1 and 3 of the Hachemeister data set into one cohort and states 2, 4 and 5 into another. This shows that data does not have to be sorted by level. The fitted model using the iterative estimators is:

```

> X <- cbind(cohort = c(1, 2, 1, 2, 2), hachemeister)
> fit <- cm(~cohort + cohort:state, data = X, ratios = ratio.1:ratio.12,
+         weights = weight.1:weight.12, method = "iterative")
> fit

```

```

Call:
cm(formula = ~cohort + cohort:state, data = X, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, method = "iterative")

```

Structure Parameters Estimators

```

Collective premium: 1746

Between cohort variance: 88981
Within cohort/Between state variance: 10952
Within state variance: 139120026

```

The function returns a fitted model object of class "cm" containing the estimators of the structure parameters. To compute the credibility premiums, one calls a method of predict for this class:

```

> predict(fit)

$cohort
[1] 1949 1543

$state
[1] 2048 1524 1875 1497 1585

```

One can also obtain a nicely formatted view of the most important results with a call to summary:

```

> summary(fit)

Call:
cm(formula = ~cohort + cohort:state, data = X, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, method = "iterative")

```

Structure Parameters Estimators

```

Collective premium: 1746

Between cohort variance: 88981
Within cohort/Between state variance: 10952
Within state variance: 139120026

```

Detailed premiums

```

Level: cohort
  cohort  Individ. mean Weight Cred. factor Cred. premium
1      1967           1.407 0.9196           1949

```

```

2      1528      1.596  0.9284      1543

```

```

Level: state

```

```

  cohort state Individ. mean Weight Cred. factor Cred. premium
1      1      2061      100155 0.8874      2048
2      2      1511      19895 0.6103      1524
1      3      1806      13735 0.5195      1875
2      4      1353      4152 0.2463      1497
2      5      1600      36110 0.7398      1585

```

The methods of `predict` and `summary` can both report for a subset of the levels by means of an argument `levels`. For example:

```

> summary(fit, levels = "cohort")

```

```

Call:

```

```

cm(formula = ~cohort + cohort:state, data = X, ratios = ratio.1:ratio.12,
  weights = weight.1:weight.12, method = "iterative")

```

```

Structure Parameters Estimators

```

```

Collective premium: 1746

```

```

Between cohort variance: 88981

```

```

Within cohort variance: 10952

```

```

Detailed premiums

```

```

Level: cohort

```

```

  cohort Individ. mean Weight Cred. factor Cred. premium
1      1967      1.407  0.9196      1949
2      1528      1.596  0.9284      1543

```

```

> predict(fit, levels = "cohort")

```

```

$cohort

```

```

[1] 1949 1543

```

The results above differ from those of [Goovaerts and Hoogstad \(1987\)](#) for the same example because the formulas for the credibility premiums are different.

4 Bühlmann and Bühlmann–Straub models

As mentioned above, the Bühlmann and Bühlmann–Straub models are simply one-level hierarchical models. In this case, the Bühlmann–Gisler and Ohlsson estimators of the between variance parameters are both identical to the usual [Bühlmann and Straub \(1970\)](#) estimator

$$\hat{a} = \frac{w_{\Sigma\Sigma}}{w_{\Sigma\Sigma}^2 - \sum_{i=1}^I w_{i\Sigma}^2} \left(\sum_{i=1}^I w_{i\Sigma} (X_{i\omega} - X_{\omega\omega})^2 - (I-1)\hat{s}^2 \right), \quad (11)$$

and the iterative estimator

$$\bar{a} = \frac{1}{I-1} \sum_{i=1}^I z_i (X_{iw} - X_{zw})^2 \quad (12)$$

is better known as the Bichsel–Straub estimator.

To fit the Bühlmann model using `cm`, one simply does not specify any weights:

```
> cm(~state, hachemeister, ratios = ratio.1:ratio.12)
```

Call:

```
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12)
```

Structure Parameters Estimators

```
Collective premium: 1671
```

```
Between state variance: 72310
```

```
Within state variance: 46040
```

In comparison, the results for the Bühlmann–Straub model using the Bichsel–Straub estimator are:

```
> cm(~state, hachemeister, ratios = ratio.1:ratio.12,  
+   weights = weight.1:weight.12)
```

Call:

```
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12,  
   weights = weight.1:weight.12)
```

Structure Parameters Estimators

```
Collective premium: 1684
```

```
Between state variance: 89639
```

```
Within state variance: 139120026
```

5 Regression model of Hachemeister

The regression model of [Hachemeister \(1975\)](#) is a generalization of the Bühlmann–Straub model. If data shows a systematic trend, the latter model will typically under- or over-estimate the true premium of an entity. The idea of [Hachemeister](#) was to fit to the data a regression model where the parameters are a credibility weighted average of an entity’s regression parameters and the group’s parameters.

In order to use `cm` to fit a credibility regression model to a data set, one simply has to supply as additional arguments `regformula` and `regdata`. The

first one is a formula of the form \sim terms describing the regression model and the second is a data frame of regressors. That is, arguments `regformula` and `regdata` are in every respect equivalent to arguments `formula` and `data` of `lm`, with the minor difference that `regformula` does not need to have a left hand side (and is ignored if present). For example, fitting the model

$$X_{it} = \beta_0 + \beta_1 t + \varepsilon_t, \quad t = 1, \dots, 12$$

to the original data set of [Hachemeister \(1975\)](#) is done with

```
> fit <- cm(~state, hachemeister,
+         regformula = ~ time, regdata = data.frame(time = 1:12),
+         ratios = ratio.1:ratio.12, weights = weight.1:weight.12)
> fit
```

Call:

```
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, regformula = ~time, regdata = data.frame(time = 1:12))
```

Structure Parameters Estimators

```
Collective premium: 1469 32.05

Between state variance: 24154 2700.0
                       2700 301.8

Within state variance: 49870187
```

Computing the credibility premiums requires to give the “future” values of the regressors as in `predict.lm`:

```
> predict(fit, newdata = data.frame(time = 13))
```

```
[1] 2437 1651 2073 1507 1759
```

It is well known that the basic regression model has a major drawback: there is no guarantee that the credibility regression line will lie between the collective and individual ones. This may lead to grossly inadequate premiums, as [Figure 1](#) shows.

The solution proposed by [Bühlmann and Gisler \(1997\)](#) is simply to position the intercept at the barycenter of time instead of at time origin (see also [Bühlmann and Gisler, 2005](#), Section 8.4). In mathematical terms, this essentially amounts to using an orthogonal design matrix. By setting the argument `adj.intercept` to `TRUE` in the call, `cm` will automatically fit the credibility regression model with the intercept at the barycenter of time. The resulting regression coefficients have little meaning, but the predictions are sensible:

```
> fit2 <- cm(~state, hachemeister,
+         regformula = ~ time, regdata = data.frame(time = 1:12),
+         adj.intercept = TRUE,
+         ratios = ratio.1:ratio.12, weights = weight.1:weight.12)
> summary(fit2, newdata = data.frame(time = 13))
```

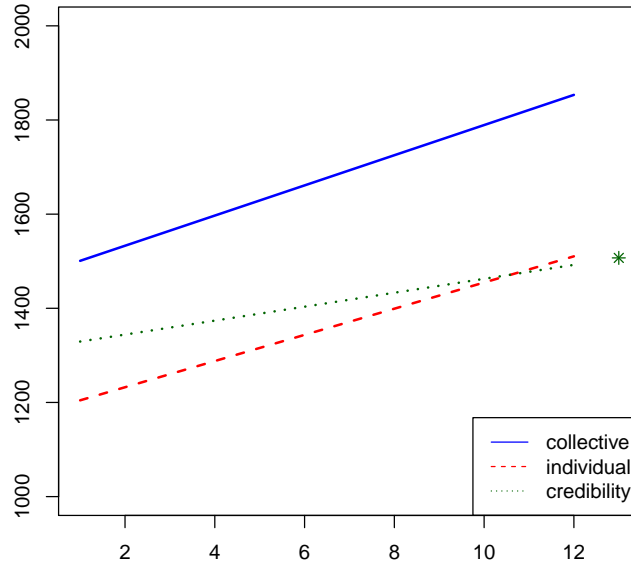



Figure 1: Collective, individual and credibility regression lines for State 4 of the Hachemeister data set. The point indicates the credibility premium.

```
Call:
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12,
  weights = weight.1:weight.12, regformula = ~time, regdata = data.frame(time = 1:12),
  adj.intercept = TRUE)
```

Structure Parameters Estimators

```
Collective premium: -1675 117.1

Between state variance: 93783 0
                       0 8046
Within state variance: 49870187
```

Detailed premiums

```
Level: state
state Individ. coef. Credibility matrix Adj. coef. Cred. premium
1     -2062.46      0.9947 0.0000      -2060.41      2457
      216.97      0.0000 0.9413          211.10
```

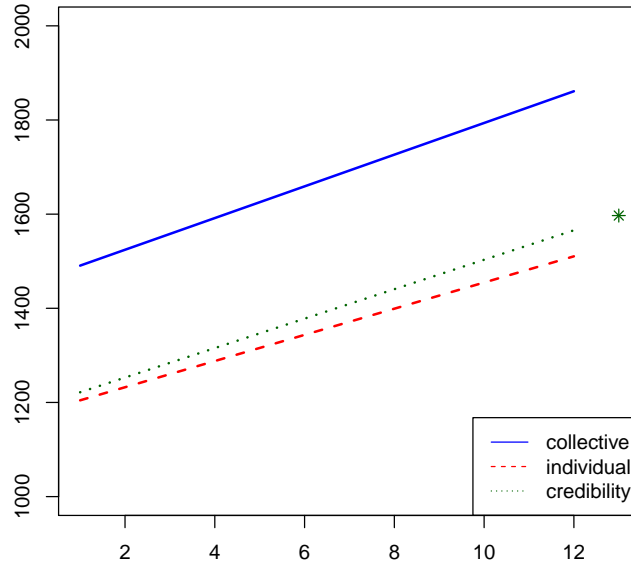


Figure 2: Collective, individual and credibility regression lines for State 4 of the Hachemeister data set when the intercept is positioned at the barycenter of time. The point indicates the credibility premium.

2	-1509.28	0.9740	0.0000	-1513.59	1651
	59.60	0.0000	0.7630	73.23	
3	-1813.41	0.9627	0.0000	-1808.25	2071
	150.60	0.0000	0.6885	140.16	
4	-1356.75	0.8865	0.0000	-1392.88	1597
	96.70	0.0000	0.4080	108.77	
5	-1598.79	0.9855	0.0000	-1599.89	1698
	41.29	0.0000	0.8559	52.22	

Figure 2 shows the beneficial effect of the intercept adjustment on the premium of State 4.

References

H. Belhadj, V. Goulet, and T. Ouellet. On parameter estimation in hierarchical credibility. *ASTIN Bulletin*, 39(2), 2009.

- H. Bühlmann. Experience rating and credibility. *ASTIN Bulletin*, 5:157–165, 1969.
- H. Bühlmann and A. Gisler. Credibility in the regression case revisited. *ASTIN Bulletin*, 27:83–98, 1997.
- H. Bühlmann and A. Gisler. *A course in credibility theory and its applications*. Springer, 2005. ISBN 3-5402575-3-5.
- H. Bühlmann and W. S. Jewell. Hierarchical credibility revisited. *Bulletin of the Swiss Association of Actuaries*, 87:35–54, 1987.
- H. Bühlmann and E. Straub. Glaubwürdigkeit für Schadensätze. *Bulletin of the Swiss Association of Actuaries*, 70:111–133, 1970.
- M. J. Goovaerts and W. J. Hoogstad. *Credibility theory*. Number 4 in Surveys of actuarial studies. Nationale-Nederlanden N.V., Netherlands, 1987.
- V. Goulet. Principles and application of credibility theory. *Journal of Actuarial Practice*, 6:5–62, 1998.
- C. A. Hachemeister. Credibility for regression models with application to trend. In *Credibility, theory and applications*, Proceedings of the Berkeley Actuarial Research Conference on Credibility, pages 129–163, New York, 1975. Academic Press.
- W. S. Jewell. The use of collateral data in credibility theory: a hierarchical model. *Giornale dell'Istituto Italiano degli Attuari*, 38:1–16, 1975.
- E. Ohlsson. Simplified estimation of structure parameters in hierarchical credibility. Presented at the Zurich ASTIN Colloquium, 2005. URL <http://www.actuaries.org/ASTIN/Colloquia/Zurich/Ohlsson.pdf>.