AMPLITUDE DOMAIN-FEQUENCY REGRESSION

Francisco Parra

March 26, 2015

Introduction

The time series can be seen from an aplitude-time domain or an amplitude-frequency domain. The amplitude-frecuency domain are used to analyze properties of filters used to decompose a time series into a trend, seasonal and irregular component investigating the gain function to examine the effect of a filter at a given frequency on the amplitude of a cycle for a particular time series. The ability to decompose data series into different frequencies for separate analysis and later recomposition is the first fundamental concept in the use of spectral techniques in forecasting, such as regression espectrum band, have had little development in econometric work. The low diffusion of this technique has been associated with the computing difficulties caused the need to work with complex numbers, and inverse Fourier transform in order to convert everything back into real terms. But the problems from the use of the complex Fourier transform may be circumvented by carrying out the Fourier transform of the data in real terms, pre-multiplied the time series by the orthogonal matrix Z whose elements are defined in Harvey (1978).

The spectral analysis commences with the assumption that any series can be transformed into a set of sine and cosine waves, and can be used to both identify and quantify apparently nonperiodic short and long cycle processes (first section). In Band spectrum regression (second section) , is a brief summary of the regression of the frequency domain (Engle, 1974) The application of spectral analysis to data containing both seasonal (high frequency) and non-seasonal (low frequency) components may produce adventages, since these different frequencies can be modelled separately and then may be re-combined to produce fitted values. Durbin (1967 and 1969) desing a technique for studying the general nature of the serial dependence in a satacionary time series, that can be use to statistic contraste in This type of exercises (third section). The time-varying regression, or the regression whit the vector of parameters time.varying can be understood in this context (four section).

Spectral analysis

Nerlove (1964) and Granger (1969) were the two foremost researchers on the application of spectral techniques to economic time series.

The use of spectral analysis requires a change of focus from an amplitude-time domain to an amplitude-frequency domain. Thus spectral analysis commences with the assumption that any series, Xt, can be transformed into a set of sine and cosine waves such as:

$$X_t = \eta + \sum_{j=1}^{N} [a_j \cos(2\pi \frac{ft}{n}) + b_j \sin(2\pi \frac{ft}{n})] \tag{1}$$

where η is the mean of the series, a_j and b_j are the amplitude, f is the frequency over a span of n observations, t is a time index ranging from 1 to N where N is the number of periods for which we have observations, the fraction (ft/n) for different values of t converts the discrete time scale of time series into a proportion of 2 and j ranges from 1 to n where n= N/2. The highest observable frequency in the series is n/N (i.e., 0.5 cycles per time interval). High frequency dynamics (large f) are akin to short cycle processes while low frequency dynamics (small f) may be likened to long cycle processes. If we let $\frac{ft}{n} = w$ then equation (1) can be re-written more compactly as:

$$X_t = \eta + \sum_{j=1}^{N} [a_j \cos(\omega_j) + b_j \sin(\omega_j)] \tag{2}$$

Spectral analysis can be used to both identify and quantify apparently non-periodic short and long cycle processes. A given series X_t may contain many cycles of different frequencies and amplitudes and such combinations of frequencies and amplitudes may yield cyclical patterns which appear non-periodic with irregular amplitude. In fact, in such a time series it is clear from equation (2) that each observation can be broken down into component parts of different length cycles which, when added together (along with an error term), comprise the observation (Wilson and Perry, 2004).

The overall effect of the Fourier analysis of N observation to a time date is to partition the variability of the series into components at frequencies $\frac{2\pi}{N}$, $\frac{4\pi}{N}$,...,π.The component at frequency $\omega_p = \frac{2\pi p}{N}$ if called the pth harmonic. For $p \neq \frac{N}{2}$, the equivalent form to write the pth harmonic are:

$$a_p cos\omega_p t + b_p sin\omega_p t = R_p cos(\omega_p t + \phi_p)$$

.

where $R_p = \sqrt{a_p + b_p}$ and $\phi_p = tan^{-1}(\frac{-b_p}{a_p})$

The plot of $I(\omega) = \frac{NR_p^2}{4\pi}$ against ω is called the periodogram of time data. Trend will produce a peak at zero frequency, while seasonal variations produces peaks at the seasonal frquency and at integer multiples of the sesaonal frequency. Then, when a periodogram has a large peak at some frequency ω then related peaks may occurr at 2ω, 3ω,....(Chaftiel, C,2004)

Band spectrum regression

Hannan (1963) first proposed regression analysis in the frequency domain,later examining the use of this technique in estimating distributed lag models (Hannan, 1965, 1967). Engle (1974) demonstrated that regression in the frequency domain has certain advantages over regression in the time domain. Consider the linear regression model

$$y = X\beta + u \tag{3}$$

where X is an n x k matrix of fixed observations on the independent variables, β is a k x I vector of parameters, y is an n x 1 vector of observations on the dependent variable, and u is an n x I vector of disturbance terms each with zero mean and constant variance, σ^2.

The model may be expressed in terms of frequencies by applying a finite Fourier transform to the dependent and independent variables.For Harvey (1978) there are a number of reasons for doing this. One is to permit the application of the technique known as 'band spectrum regression', in which regression is carried out in the frequency domain with certain wavelengths omitted. Another reason for interest in spectral regression is that if the disturbances in (3) are serially correlated, being generated by any stationary stochastic process, then regression in the frequency domain will yield an asymptotically efficient estimator of β.

Engle (1974) compute the full spectrum regression with he complex finite Fourier transform based on the n x n matrix W, in which element (t, s) is given by

$w_{ts} = \frac{1}{\sqrt{n}} e^{i\lambda_t s}$, $s = 0, 1, ..., n - 1$

where $\lambda_t = 2\pi\frac{t}{n}$, t=0,1,...,n-1, and $i = \sqrt{-1}$.

Pre-multiplying the observations in observations in (3) by W yields

$$\dot{y} = \dot{X}\beta + \dot{u} \tag{4}$$

where $\dot{y} = Wy$,$\dot{X} = WX$, and $\dot{u} = Wu$.

If the disturbance vector in (4) obeys the classical assumptions, viz. $E[u] = 0$ and $E[uu'] = \sigma^2 I_n$. then the transformed disturbance vector, $\dot{u}$, will have identical properties. This follows because the matrix W is unitary, i.e., $WW^T = I$, where W^T is the transpose of the complex conjugate of W. Furthermore the observations in (4) contain precisely the same amount of information as the untransformed observations in (3).

Application of OLS to (4) yields, in view of the properties of $\dot{u}$, the best linear unbiased estimator (BLUE) of β. This estimator is identical to the OLS estimator in (3), a result which follows directly on taking account of the unitary property of W. When the relationship implied by (4) is only assumed to hold for certain frequencies, band spectrum regression is appropriate, and this may be carried out by omitting the observations in (4) corresponding to the remaining frequencies. Since the variables in (4) are complex, however, Engle (1974) suggests an inverse Fourier transform in order to convert everything back into real terms (Harvey,1974).

The problems which arise from the use of the complex Fourier transform may be circumvented by carrying out the Fourier transform of the data in real terms.

In order to do this the observations in (3) are pre-multiplied by the orthogonal matrix Z whose elements are defined as follows (Harvey,1978):

$$z_{ts} = \begin{cases} \left(\frac{1}{n}\right)^{\frac{-1}{2}} & t = 1 \\ \left(\frac{2}{n}\right)^{\frac{1}{2}} \cos\left[\frac{\pi t(s-1)}{n}\right] & t = 2,4,6,..(n-2) \text{ or } (n-1) \\ \left(\frac{2}{n}\right)^{\frac{1}{2}} \sin\left[\frac{\pi(t-1)(s-1)}{T}\right] & t = 3,5,7,..,(n-1) \text{ or } n \\ (n)^{\frac{-1}{2}}(-1)^{s+1} & t = n \text{ if n is even }, s = 1,...n, \end{cases}$$

The resulting frequency domain regression model is:

$$y^{**} = X^{**}\beta + v \tag{5}$$

where $y^{**} = Zy$,$X^{**} = ZX$ and $v = Zu$.

In view of the orthogonality of Z, $E[vv'] = \sigma^2 I_n$ when $E[uu'] = \sigma^2 I_n$ and the application of OLS to (5) gives the BLUE of β.

Since all the elements of y** and X** are real, model may be treated by a standard regression package. If band spectrum regression is to be carried out, the number of rows in y** and X** is reduced accordingly, and so no problems arise from the use of an inappropriate number of degrees of freedom.

Amplitude domain-frequency regression

Consider now the linear regression model

$$y_t = \beta_t x_t + u_t \tag{6}$$

where x_t is an n x 1 vector of fixed observations on the independent variable, β_t is a n x 1 vector of parameters,y is an n x 1 vector of observations on the dependent variable, and u_t is an n x 1 vector de errores distribuidos con media cero y varianza constante.

Whit the assumption that any series, y_t,x_t,β_t and ut, can be transformed into a set of sine and cosine waves such as:

$$y_t = \eta^y + \sum_{j=1}^{N} [a_j^y \cos(\omega_j) + b_j^y \sin(\omega_j)$$

$$x_t = \eta^x + \sum_{j=1}^{N} [a_j^y \cos(\omega_j) + b_j^y \sin(\omega_j)]$$

$$\beta_t = \eta^\beta + \sum_{j=1}^{N} [a_j^\beta \cos(\omega_j) + b_j^\beta \sin(\omega_j)]$$

Pre-multiplying (6) by Z:

$$\dot{y} = \dot{x}\dot{\beta} + \dot{u}$$

(7)

where $\dot{y} = Zy$,$\dot{x} = Zx$, $\dot{\beta} = Z\beta$ y $\dot{u} = Zu$

The system (7) can be rewritten as (see appendix):

$$\dot{y} = Zx_t I_n Z^T \dot{\beta} + Z I_n Z^T \dot{u}$$

(8)

If we call $\dot{e} = Z I_n Z^T \dot{u}$, It can be found the $\dot{\beta}$ that minimize the sum of squared errors $E_T = Z^T \dot{e}$.

Once you have found the solution to this optimization, the series would be transformed into the time domain.

Seasonal Decomposition by the Fourier Coefficients

The amplitude domain-frequency regression method could be use to decompose a time series into seasonal, trend and irregular components of a time serie y_t of frequency b or number of times in each unit time interval. For example, one could use a value of 7 for frequency when the data are sampled daily, and the natural time period is a week, or 4 and 12 when the data are sampled quarterly and monthly and the natural time period is a year.

If the observation are teken at equal interval of length, $\triangle t$, then the angular frequency is $\omega = frac\pi\triangle t$. The equivalent frequency expressed in cycles per unit time is $f = \frac{\omega}{2\pi} = \frac{1}{2} \triangle t$. Whit only one observation per year, $\omega = \pi$ radians per year or $f = \frac{1}{2}$ cycle per year (1 cicle per two years), variation whit a wavelength of one year has fequency $\omega = 2\pi$ radians per year or $f = 1$ cicle per year.

For example, in a monthly time serie of $N = 100$ observation, the seasonal cycles or the wavelenghth of one year has frequency $f = \frac{100}{12} = 8,33$ cycles for 100 dates. If the time serie are 8 full year, the less seasonal frequency are 1 cycle for year, or 8 cycle for 96 observation. The integer multiplies are $2\frac{N}{12}$,$3\frac{N}{12}$...., and wavelenghth low of one year has frequency are $f < \frac{N}{12}$.

We can use (8) to estimate the fourier coefficient in time serie y_t:

$$\dot{y} = Zt I_n Z^T \dot{\beta} + Z I_n Z^T \dot{u}$$

(9)

being $t = (1, 1, ...1)_N$ or $t = (1, 2, 3, ..., N)_N$.

If $t = (1, 1, 1, ...1)_N$,

$$A = Zt I_n Z^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & 0 & 1 \end{pmatrix}$$

Then

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & 0 & 0 \end{pmatrix}$$

are use in (9) to make the regression band spectrum with the first four coefficient of fourier of the serie $\dot{y}$.

The first$2\frac{N}{12}-1$ rows the A matrix are used to estimate the fourier coefficients corresponding to cycles of low frequency, trend cycles, and rows $2\frac{N}{12}$ and $2\frac{N}{12}+1$ are used to estimate the fourier coefficients of 1 cicle for year. The integer multiplies re the rows $6\frac{N}{12}$, $6\frac{N}{12}+1$, $8\frac{N}{12}$...should be used to obtain the seasonal frequency.

Example:descomponse by amplitude domain-frequency regression. IPI base 2009 in Cantabria

The Industrial Price Index of Cantabria is presented in the table below

The time serie by trend an seasonal is named $TDST$. TD is calculate by band spectrum regresion of the serie y_t and the temporal index t, in which regression is carried out in low amplitude- frequency. The seasonal serie ST result to take away TD to $TDST$, and the irregular serie IR result to take away $TDST$ to y_t (figure 8). The temporal index t used in the exemple are the OLS regression into IPI and the trend index $t = (1, 2, 3, ...N)_N$.

```
> library(descomponer)
> data(ipi)
> descomponer(ipi,12,1)$datos
```

```
       y      TDST        TD          ST           IR
1   90.2  93.49148  97.29581  -3.8043288  -3.29147706
2   98.8  96.76618  97.40651  -0.6403355   2.03382281
3   92.1 105.16011  97.55957   7.6005392 -13.06010720
4  102.7 100.11383  97.73672   2.3771122   2.58616508
5  107.0 105.36545  97.91825   7.4471960   1.63455301
6   98.3 102.67619  98.08444   4.5917463  -4.37619107
7  100.9  99.14371  98.21717   0.9265446   1.75628888
8   66.3  72.41965  98.30134 -25.8816898  -6.11964836
9  101.4 100.48346  98.32624   2.1572165   0.91654243
10 111.8 107.36550  98.28651   9.0789861   4.43450007
11 111.4 105.66091  98.18276   7.4781476   5.73909316
12  85.2  86.24833  98.02170 -11.7733676  -1.04832922
13  94.4  94.02740  97.81584  -3.7884330   0.37259602
14  96.2  96.94503  97.58269  -0.6376590  -0.74503016
15 106.5 104.91231  97.34356   7.5687593   1.58768510
16 101.1  99.48917  97.12200   2.3671694   1.61083240
```

17	103.5	104.35813	96.94209	7.4160356	-0.85812832
18	99.9	101.39913	96.82661	4.5725269	-1.49913452
19	101.4	97.71791	96.79525	0.9226651	3.68208654
20	58.6	71.08983	96.86311	-25.7732827	-12.48982901
21	99.8	99.18765	97.03947	2.1481777	0.61234851
22	112.7	106.36795	97.32701	9.0409316	6.33205472
23	103.8	105.16833	97.72154	7.4467921	-1.36833257
24	89.0	86.48826	98.21225	-11.7239851	2.51173577
25	91.2	95.00995	98.78249	-3.7725372	-3.80995442
26	97.3	98.77602	99.41100	-0.6349825	-1.47601811
27	110.2	107.61046	100.07348	7.5369794	2.58954158
28	105.7	103.10165	100.74442	2.3572266	2.59835269
29	109.9	108.78390	101.39902	7.3848751	1.11610157
30	109.1	106.56835	102.01504	4.5533075	2.53165476
31	104.3	103.49319	102.57441	0.9187856	0.80680894
32	71.9	77.39968	103.06455	-25.6648756	-5.49967709
33	107.1	105.61838	103.47924	2.1391389	1.48162425
34	108.5	112.82176	103.81888	9.0028771	-4.32175586
35	116.6	111.50579	104.09036	7.4154366	5.09420740
36	96.5	92.63167	104.30628	-11.6746027	3.86832534
37	94.1	100.72716	104.48380	-3.7566413	-6.62715880
38	102.4	104.01079	104.64309	-0.6323060	-1.61078761
39	109.4	112.31078	104.80558	7.5051995	-2.91077599
40	109.0	107.33936	104.99208	2.3472838	1.66063840
41	113.3	112.57479	105.22108	7.3537147	0.72520870
42	116.5	110.04125	105.50717	4.5340881	6.45874503
43	107.9	106.77478	105.85988	0.9149060	1.12521823
44	76.7	80.72646	106.28293	-25.5564685	-4.02646394
45	111.0	108.90415	106.77405	2.1301001	2.09585363
46	109.3	116.29003	107.32521	8.9648227	-6.99002963
47	119.5	115.30756	107.92348	7.3840810	4.19244058
48	95.1	96.92699	108.55221	-11.6252202	-1.82698928
49	109.6	105.45181	109.19255	-3.7407455	4.14819131
50	109.0	109.19553	109.82516	-0.6296296	-0.19553175
51	125.2	117.90530	110.43188	7.4734196	7.29469750
52	104.8	113.33469	110.99734	2.3373410	-8.53468571
53	123.7	118.83279	111.51024	7.3225543	4.86720973
54	119.7	116.47907	111.96420	4.5148687	3.22093379
55	105.4	113.26925	112.35822	0.9110265	-7.86924913
56	84.1	87.24847	112.69653	-25.4480614	-3.14846658
57	112.1	115.10896	112.98790	2.1210613	-3.00895781
58	121.6	122.17131	113.24454	8.9267682	-0.57131077
59	120.0	120.83332	113.48059	7.3527255	-0.83331896
60	98.6	102.13448	113.71031	-11.5758378	-3.53447654
61	117.6	110.22138	113.94623	-3.7248497	7.37861665
62	117.7	113.57038	114.19733	-0.6269531	4.12962237

63	129.7	121.90910	114.46747	7.4416398	7.79089518
64	111.8	117.08157	114.75418	2.3273982	-5.28157443
65	125.2	122.33939	115.04800	7.2913939	2.86060751
66	121.2	119.82801	115.33236	4.4956493	1.37198834
67	116.8	116.49127	115.58412	0.9071470	0.30873208
68	88.2	90.43504	115.77469	-25.3396543	-2.23503871
69	113.7	117.98378	115.87175	2.1120225	-4.28377603
70	129.0	124.73008	115.84137	8.8887137	4.26992094
71	121.7	122.97177	115.65040	7.3213700	-1.27177389
72	94.4	103.74264	115.26910	-11.5264553	-9.34264377
73	110.3	110.96455	114.67351	-3.7089538	-0.66455342
74	115.3	113.22345	113.84773	-0.6242766	2.07655123
75	112.9	120.19554	112.78568	7.4098599	-7.29553587
76	122.4	113.80977	111.49232	2.3174554	8.59022509
77	116.9	117.24442	109.98419	7.2602334	-0.34442458
78	111.2	112.76564	108.28921	4.4764299	-1.56563785
79	115.0	107.34901	106.44574	0.9032674	7.65098972
80	77.1	79.26977	104.50102	-25.2312472	-2.16976916
81	106.3	104.61189	102.50890	2.1029837	1.68811145
82	115.9	109.37796	100.52731	8.8506592	6.52203544
83	106.7	105.90524	98.61523	7.2900144	0.79475657
84	83.0	85.35274	96.82981	-11.4770729	-2.35273788
85	92.2	91.53037	95.22343	-3.6930580	0.66962853
86	94.3	93.21952	93.84112	-0.6216001	1.08048013
87	96.7	100.09652	92.71844	7.3780800	-3.39651790
88	87.2	94.18741	91.87990	2.3075126	-6.98741330
89	91.0	98.56716	91.33809	7.2290730	-7.56716185
90	91.0	95.55065	91.09344	4.4572105	-4.55065228
91	95.3	92.03412	91.13474	0.8993879	3.26587643
92	70.2	66.31734	91.44018	-25.1228401	3.88265582
93	98.3	94.07301	91.97906	2.0939449	4.22699051
94	106.9	101.52634	92.71374	8.8126048	5.37365804
95	103.4	100.86057	93.60191	7.2586589	2.53942747
96	86.8	83.17132	94.59901	-11.4276905	3.62867995
97	90.5	91.98328	95.66044	-3.6771622	-1.48327720
98	91.4	96.12477	96.74369	-0.6189236	-4.72476795
99	107.7	105.15641	97.81010	7.3463001	2.54359493
100	100.6	101.12380	98.82623	2.2975698	-0.52379809
101	101.9	106.96265	99.76474	7.1979126	-5.06264944
102	105.8	105.04288	100.60489	4.4379911	0.75712158
103	101.5	102.22804	101.33254	0.8955084	-0.72804413
104	75.4	76.92534	101.93977	-25.0144330	-1.52533899
105	101.4	104.50915	102.42425	2.0849062	-3.10915268
106	109.1	111.56283	102.78828	8.7745503	-2.46283178
107	115.8	110.26517	103.03786	7.2273034	5.53483418
108	98.9	91.80330	103.18160	-11.3783080	7.09670316

```
109  97.6  99.56851 103.22978  -3.6612663  -1.96851275
110 102.7 102.57721 103.19346  -0.6162472   0.12278761
111 113.2 110.39836 103.08384   7.3145202   2.80163645
112 104.3 105.19939 102.91176   2.2876270  -0.89938650
113 107.6 109.85412 102.68737   7.1667522  -2.25412104
114 103.5 106.83880 102.42003   4.4187717  -3.33880379
115  97.9 103.00994 102.11831   0.8916288  -5.10993901
116  86.3  76.88402 101.79005 -24.9060259   9.41597537
117 108.4 103.51838 101.44251   2.0758674   4.88162432
118 103.5 109.81895 101.08246   8.7364958  -6.31895228
119 103.5 107.91219 100.71625   7.1959478  -4.41219319
120  89.0  89.02086 100.34979 -11.3289256  -0.02086087
121  94.5  96.34308  99.98845  -3.6453705  -1.84307991
122  97.7  99.02332  99.63689  -0.6135707  -1.32331522
123 112.9 106.58152  99.29878   7.2827404   6.31847874
124  97.6 101.25429  98.97660   2.2776842  -3.65428531
125 111.6 105.80694  98.67135   7.1355917   5.79306067
126 103.8 102.78193  98.38238   4.3995523   1.01806936
127  97.3  98.99509  98.10734   0.8877493  -1.69508506
128  86.6  73.04459  97.84221 -24.7976188  13.55540629
129  94.7  99.64840  97.58158   2.0668286  -4.94840385
130 100.3 106.01739  97.31895   8.6984413  -5.71739065
131  95.4 104.21194  97.04735   7.1645923  -8.81193975
132  85.4  85.48037  96.75991 -11.2795431  -0.08036676
133  96.3  92.82113  96.45061  -3.6294747   3.47886911
134  94.5  95.50404  96.11493  -0.6108942  -1.00403907
135  98.1 103.00152  95.75056   7.2509605  -4.90151667
136 105.0  97.62554  95.35780   2.2677414   7.37445589
137 101.0 102.04441  94.93997   7.1044313  -1.04440606
138  98.8  98.88375  94.50342   4.3803329  -0.08375036
139  91.5  94.94119  94.05732   0.8838697  -3.44119438
140  80.5  68.92406  93.61328 -24.6892117  11.57593655
141  94.6  95.24231  93.18452   2.0577898  -0.64231218
142 100.6 101.44547  92.78508   8.6603868  -0.84546802
143  91.8  99.56192  92.42868   7.1332368  -7.76191538
144  82.1  80.89749  92.12765 -11.2301607   1.20250882
145  91.8  88.08756  91.89189  -3.8043288   3.71244372
146  92.6  91.08754  91.72788  -0.6403355   1.51245564
147 100.1  99.23859  91.63805   7.6005392   0.86141476
148  95.4  93.99740  91.62029   2.3771122   1.40259993

> gdescomponer(ipi,12,1,2002,1)
```

Bibliography

Chatfield, Cris (2004). "The Analysis of Time Series: An Introduction (6th edn.)", 2004. CRC Press

Engle, Robert F. (1974), "Band Spectrum Regression", International Economic Review 15,1-11.

Hannan, E.J. (1963), "Regression for Time Series", in Rosenblatt, M. (ed.), Time Series Analysis, New York, John Wiley.

Harvey, A.C. (1978), "Linear Regression in the Frequency Domain", International Economic Review, 19, 507-512.

Wilson, P.J. and Perry, L.J. (2004). "Forecasting Australian Unemployment Rates Using Spectral Analysis" Australian Jurnal of Labour Economics, vol 7,no 4, December 2004, pp 459-480.

Appendix

The multiplication of two harmonic series of diferent frequency:

$$[a_j \cos(\omega_j) + b_j \sin(\omega_j)]x[a_i \cos(\omega_i) + b_i \sin(\omega_i)]$$

gives the following sum:

$$a_j a_i \cos(\omega_j) \cos(\ omega_i) + a_j b_i \cos(\omega_j) \sin(\omega_i)$$

$$+a_i b_j \sin(\omega_j) \cos(\omega_i) b_i \sin(\omega_i) + b_j b_i \sin(\omega_j) \sin(\omega_i)$$

that using the identity of the products of sines and cosines gives the following results:

$$\frac{a_j a_i + b_j b_i}{2} \cos(\omega_j - \omega_i) + \frac{b_j a_i - b_j a_i}{2} \sin(\omega_j - \omega_i)$$

$$+\frac{a_j a_i - b_j b_i}{2} \cos(\omega_j + \omega_i) + + \frac{b_j a_i + b_j a_i}{2} \sin(\omega_j + \omega_i)$$

The circularity of ω determines that the product of two harmonics series resulting in a new series in which the Fourier coefficients it's a linear combination of the Fourier coefficients of the two harmonics series.

In the following two series:

$$y_t = \eta^y + a_0^y \cos(\omega_0) + b_0^y \sin(\omega_0) + a_1^y \cos(\omega_1) + b_1^y \sin(\omega_1) + a_2^y \cos(\omega_2) + b_2^y \sin(\omega_2) + a_3^y \cos(\omega_3)$$

$$x_t = \eta^x + a_0^x \cos(\omega_0) + b_0^x \sin(\omega_0) + a_1^x \cos(\omega_1) + b_1^x \sin(\omega_1) + a_2^x \cos(\omega_2) + b_2^x \sin(\omega_2) + a_3^x \cos(\omega_3)$$

given a matrix $\Theta^{\dot{x}\dot{x}}$ of size 8x8 :

$$\Theta^{\dot{x}\dot{x}} = \eta^x I_8 + \frac{1}{2} \begin{pmatrix} 0 & a_0^x & b_0^x & a_1^x & b_1^x & a_2^x & b_2^x & 2a_3^x \\ 2a_0^x & a_1^x & b_1^x & a_0^x + a_2^x & b_0^x + b_2^x & a_1^x + 2a_3^x & b_1^x & 2a_2^x \\ 2b_0^x & b_1^x & -a_1^x & -b_0^x + b_2^x & a_0^x - a_2^x & -b_1^x & a_1^x - a_3^x & -2b_2^x \\ 2a_1^x & a_0^x + a_2^x & -b_0^x + b_2^x & 2a_3^x & 0 & a_0^x + a_2^x & b_0^x - b_2^x & 2a_1^x \\ 2b_1^x & a_0^x + b_2^x & -b_0^x - a_2^x & 0 & -2a_3^x & -b_0^x + b_2^x & a_0^x - a_2^x & -2b_1^x \\ 2a_2^x & a_1^x + 2a_3^x & -b_1^x & a_0^x + a_2^x & -b_0^x - b_2^x & a_1^x & -b_1^x & 2a_0^x \\ 2b_2^x & b_1^x & a_1^x - 2a_3^x & b_0^x - b_2^x & a_0^x - a_2^x & -b_1^x & -a_1^x & -2b_0^x \\ 2a_3^x & a_2^x & -b_2^x & a_1^x & -b_1^x & a_0^x & -b_0^x & 0 \end{pmatrix}$$

Demonstrates that:

$$\dot{z} = \Theta^{\dot{x}\dot{x}} \dot{y}$$

where $\dot{y} = Wy$,$\dot{x} = Wx$, and $\dot{z} = Wz$.

$$z_t = x_t y_t = W^T \dot{x} W^T \dot{y} = W^T W x_t W^T \dot{y} = x_t I_n W^T \dot{y}$$

$$W^T \dot{z} = x_t I_n W^T \dot{y}$$

$$\dot{z} = W^T x_t I_n W \dot{y}$$

It is true that;

$$x_t I_n = W^T \Theta^{\dot{x}\dot{x}} W$$

and

$$\Theta^{\dot{x}\dot{x}} = W^T x_t I_n W$$