

trackeR: Infrastructure for Running and Cycling Data from GPS-Enabled Tracking Devices in R

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Abstract

The use of GPS-enabled tracking devices and heart rate monitors is becoming increasingly common in sports and fitness activities. The **trackeR** package aims to fill the gap between the routine collection of data from such devices and their analyses in a modern statistical environment like R. The package provides methods to read tracking data and store them in session-based, unit-aware, and operation-aware objects of class “**trackeRdata**”. The package also implements core infrastructure for relevant summaries and visualisations, as well as support for handling units of measurement. There are also methods for relevant analytic tools such as time spent in zones, work capacity above critical power (known as W'), and distribution and concentration profiles. A case study illustrates how the latter can be used to summarise the information from training sessions and use it in more advanced statistical analyses.

Keywords: sports, tracking, work capacity, running, cycling, distribution profiles.

1. Introduction

Recent technological advances allow the collection of detailed data on fitness activities and on multiple aspects of training and competition in professional sport. The focus of this paper is on data collected by GPS-enabled tracking devices and heart rate monitors. Such devices are routinely used in fitness activities such as running and cycling, and also during training in sports like field hockey and football. Basic questions associated with tracking data include how often, much, or hard an individual or a group trains, and a more advanced outlook tries to explain the impact of training on athlete physiology or performance.

Tools for basic analytics are usually offered by the manufacturers of the tracking devices, such as Garmin, Polar, and Catapult, and through a wide range of applications for devices such as smartphones and smartwatches, e.g., Strava Running and Cycling GPS, Endomondo – Running & Walking, and Runtastic Running GPS Tracker. A notable open-source effort is Golden Cheetah (<http://www.goldencheetah.org>), which has now, perhaps, become the gold standard in terms of facilities for importing tracking data from cycling activities and for associated analytics. However, Golden Cheetah is not designed to offer general flexibility in the statistical analysis of such sports tracking data.

The R system for statistical computing (R Core Team 2015) with its ecosystem of add-on packages provides a wide range of possibilities for the analysis of tracking data. For example, mixed effects regression modelling (e.g., **lme4**, Bates, Mächler, Bolker, and Walker

2015), clustering (e.g., **mclust**, Fraley, Raftery, Murphy, and Scrucca 2012; Fraley and Raftery 2002), time series and changepoint analysis (e.g., **forecast** and **strucchange**, Hyndman and Khandakar 2008; Zeileis, Leisch, Hornik, and Kleiber 2002, respectively), methods for spatial statistics (e.g., **sp**, Pebesma and Bivand 2005), and functional data analysis (e.g., **fda**, Ramsay, Wickham, Graves, and Hooker 2014) are all relevant to data analytic tasks in the context of sport and are supported by comprehensive and actively developed packages in the R ecosystem.

Despite this wide range of R packages, there is only a handful of packages specific to sport data and their analysis. The available packages focus on topics such as sports management (**RcmdrPlugin.SM**, Champely 2012), ranking sports teams (**mvglmmRank**, Karl and Broatch 2015), and accessing betting odds (**pinnacle.API**, Blume, Jhirad, and Gassem 2015). **SportsAnalytics** is a package that focuses on the analysis of performance data, and currently offers only “a selection of data sets, functions to fetch sports data, examples, and demos” (Eugster 2013). The **cycleRtools** package (Mackie 2015) provides functionality to import cycling data into R as well as tools for cycling-specific, descriptive analyses.

The **trackeR** package aims to fill the gap between the routine collection of data from GPS-enabled tracking devices and the analyses of such data within the R ecosystem. The package provides utilities to import sports data from GPS-enabled devices, and, after careful processing, organises them in session-based, unit-aware, and operation-aware objects in R. The package also implements core infrastructure for the handling of measurement units and for summarising and visualising tracking data. It also provides functionality for calculating time in zones (e.g., Seiler and Kjerland 2006), work capacity W' (Skiba, Chidnok, Vanhatalo, and Jones 2012), and distribution and concentration profiles (Kosmidis and Passfield 2015).

Section 2 gives an overview of the package and introduces the basic objects and the methods that apply to them. Section 3 describes the importing utilities, and Section 4 details the structure and construction of the “**trackeRdata**” object, which is at the core of **trackeR**. Section 5 is devoted to the calculation of relevant summaries (time in zones, work capacity, distribution and concentration profiles) and the corresponding methods for visualisation. Section 6 and Section 7 focus on basic methods for unit manipulation, smoothing and thresholding. The case study in Section 8 investigates the key features in 26 sessions through a functional principal components analysis (e.g., Ramsay and Silverman 2005) on the concentration profiles for speed. Finally, Section 9 gives some concluding remarks and directions for further development.

2. Package structure

Figure 1 shows a schematic overview of the package structure. Squared boxes indicate objects of a particular class, and boxes with rounded corners represent methods that apply to those objects. The respective class and method names are given in the boxes. An arrow from an object to a method indicates that the method applies to objects of the respective class; an arrow from a method to an object indicates that the method outputs objects of the respective class. A bi-directional arrow between a method and an object indicates that the method’s input and output are of the same class (e.g. the method **threshold()** and objects of class “**trackeRdata**”). Arrows to or from groups of boxes apply to each box in the group (for example, the method **changeUnits()** applies to objects of classes “**trackeRdataZones**”, “**trackeRdataSummary**”, “**trackeRWprime**”, “**distrProfile**”, and “**conProfile**”).

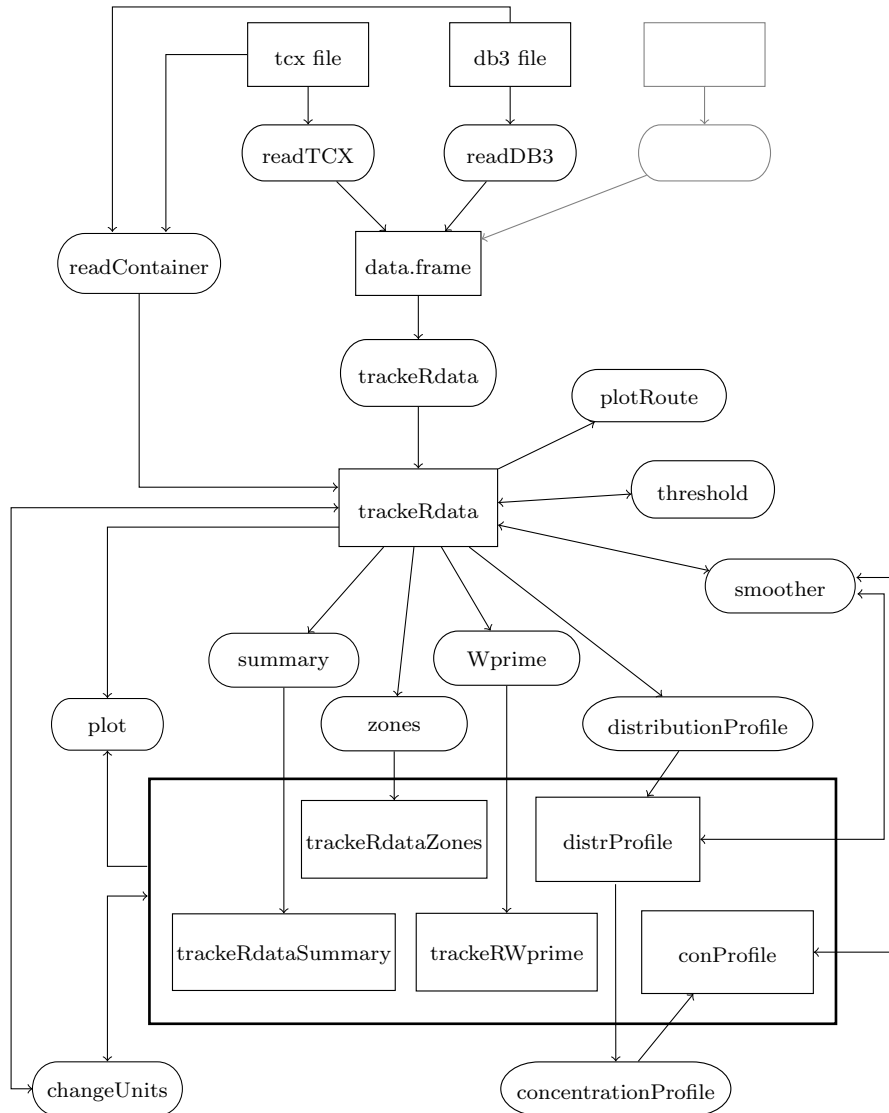


Figure 1: Package structure.

Data from various formats is imported and stored in the central data object of class “`trackeRdata`” from which summaries for descriptive purposes or further analyses can be derived. Methods for visualisation and data handling are available for data objects and summary objects. A list of all functionality is provided in Table 1.

3. Import utilities

trackeR provides utilities for data from common formats from GPS-enabled tracking devices. The family of the supplied reading functions, `read*`(), currently includes functions for reading TCX (Training Centre XML) and DB3 (for SQLite, used, e.g., by devices from GPSports) files. These functions read the tracking data, and return a `data.frame` with a specific structure. The following code chunk illustrates the use of the `readTCX`() function using a TCX file that

Function	Class	Description
c()	"trackeRdata"	combine sessions
[]	"trackeRdata"	subset sessions
plot()	"trackeRdata"	plot session profiles
plotRoute()	"trackeRdata"	plot route taken during a session
threshold()	"trackeRdata"	apply lower and upper bounds on data range
smoother()	"trackeRdata"	smooth variables by applying a summary function such as mean or median to a window
getUnits()	"trackeRdata"	access units of measurement
changeUnits()	"trackeRdata"	change units of measurement
fortify()	"trackeRdata"	convert object into a data frame for plotting
summary()	"trackeRdata"	summarise sessions
print()	"trackeRdataSummary"	print sessions summaries
getUnits()	"trackeRdataSummary"	access units of measurement
changeUnits()	"trackeRdataSummary"	change units of measurement
fortify()	"trackeRdataSummary"	convert object into a data frame for plotting
plot()	"trackeRdataSummary"	plot session summaries
zones()	"trackeRdata"	time spent in zones
getUnits()	"trackeRdataZones"	access units of measurement
changeUnits()	"trackeRdataZones"	change units of measurement
fortify()	"trackeRdataZones"	convert object into a data frame for plotting
plot()	"trackeRdataZones"	plot zone summaries
Wprime()	"trackeRdata"	calculate W' balance and W' expended
plot()	"trackeRWprime"	plot W' balance and W' expended
distributionProfile()	"trackeRdata"	calculate distribution profiles
c()	"distrProfile"	combine distribution profiles
getUnits()	"distrProfile"	access units of measurement
changeUnits()	"distrProfile"	change units of measurement
smoother()	"distrProfile"	smooth distribution profiles
fortify()	"distrProfile"	convert object into a data frame for plotting
plot()	"distrProfile"	plot distribution profiles
concentrationProfile()	"distrProfile"	calculate concentration profiles
c()	"conProfile"	combine concentration profiles
getUnits()	"conProfile"	access units of measurement
changeUnits()	"conProfile"	change units of measurement
smoother()	"conProfile"	smooth concentration profiles
fortify()	"conProfile"	convert object into a data frame for plotting
plot()	"conProfile"	plot concentration profiles

Table 1: Functions available in **trackeR** package.

ships with the package, and shows the name and type of variables that are present in the resulting data frame.

```
R> filepath <- system.file("extdata", "2013-06-08-090442.TCX", package = "trackeR")
R> ## read raw data
R> runDF <- readTCX(file = filepath, timezone = "GMT")
R> ## runDF is a data.frame with the following structure
R> str(runDF)
```

```
'data.frame':      1191 obs. of  9 variables:
 $ time      : POSIXct, format: "2013-06-08 08:04:42" ...
 $ latitude  : num  51.4 51.4 51.4 51.4 51.4 ...
 $ longitude : num  1.04 1.04 1.04 1.04 1.04 ...
 $ altitude  : num  6.2 6.2 6.2 6.2 6.2 ...
 $ distance  : num  0 1.68 5.28 8.33 14.88 ...
 $ heart.rate: num  83 84 84 86 89 93 96 98 101 102 ...
 $ speed     : num  0 0.594 1.416 1.928 2.614 ...
 $ cadence   : num  NA 54 74 97 97 97 97 98 97 97 ...
 $ power     : num  NA NA NA NA NA NA NA NA NA NA ...
```

Power is not available in the above data frame because the data come from running training. Times are taken here to be in GMT. The default for argument `timezone` is "" and is system-specific, see `?as.POSIXct` for details.

trackeR can accommodate the addition of extra formats by simply authoring appropriate import functions. Such functions should take as input the path of the file to be read and return a data frame with the same structure as in the above example.

4. trackeRdata class

4.1. Object structure

As is apparent in Figure 1, the core object has class “`trackeRdata`”. The “`trackeRdata`” objects are session-based, unit-aware and operation-aware structures, which organise the data in a list of multivariate `zoo` objects (see, [Zeileis and Grothendieck 2005](#), for a description of the `zoo` package), with one element per session. The observations within each session are ordered according to the time stamps as these are read from the GPS-enabled tracking devices. Each “`trackeRdata`” object has an attribute on the measurement units of the data it holds, and, if applicable, an attribute detailing the operations (e.g., smoothing) it has gone through.

“`trackeRdata`” objects result from the constructor function `trackeRdata()`, which takes as input the output of the `read*()` functions. Apart from the allocation of observations into distinct sessions, the constructor function also performs some data processing, including basic sanity checks (for example, removing observations with negative or missing values for cumulative distance or speed), handling of measurement units, correction of distances using altitude data if required, and data imputation.

4.2. Constructor function

The interface of the constructor function for class “`trackeRdata`” is

```
trackeRdata(dat, units = NULL, cycling = FALSE, sessionThreshold = 2,
  correctDistances = FALSE, country = NULL, mask = TRUE,
  fromDistances = TRUE, lgap = 30, lskip = 5, m = 11)
```

`dat` is the data frame containing the tracking data and `units` is used to specify the units of measurement. Table 2 shows the currently supported units and notes the units that are used by default when `units = NULL`. The argument `cycling` flags the data as coming from cycling session(s) rather than running session(s). This affects the calculation of W' (based on power or speed for cycling and running, respectively) and the thresholds applied before plotting the session data. The other arguments are specific to the data processing operations, which are briefly described in the following subsections.

4.3. Identifying distinct sessions

The constructor function groups the observations into sessions according to their time stamps. Specifically, the time stamps in the data from the `read*()` functions are first sorted, and all consecutive observations whose time stamps are no further apart from each other than a specified threshold t^* are considered to belong to a distinct session. The value of t^* is set via the `sessionThreshold` argument of the `trackeRdata()` function and it defaults to 2 hours.

4.4. Distance correction using altitude data

If the distances in the data have been calculated solely based on latitude and longitude data, without taking into account the altitude, then the distance covered can be underestimated. The `correctDistances` argument of the `trackeRdata()` function controls whether the distances should be corrected for altitude changes.

If the uncorrected distance covered at time point t_i is $d_{2,i}$, then setting `correctDistances = TRUE` uses the Pythagorean theorem to correct the distance covered between time point t_{i-1} and time point t_i to

$$d_i - d_{i-1} = \sqrt{(d_{2,i} - d_{2,i-1})^2 + (a_i - a_{i-1})^2},$$

where d_i and a_i are the corrected cumulative distance and the altitude at time t_i , respectively.

If no altitude measurements are available, then these are extracted from the *SRTM 90m Digital Elevation Data* via the `raster` package (Hijmans 2015) using the latitude and longitude measurements. The arguments `country` and `mask` control the extraction of altitudes.

4.5. Imputation process

Occasionally, there is a large time difference between consecutive observations in the same session, sometimes of the order of several minutes. This can happen, for example, if the device is intentionally paused by the athlete or if the proprietary algorithm controlling the operating sampling rate of the device detects no significant change in position. For example, in the manual of a GPS device, the Forerunner[©] 310XT, it is stated that “*The Forerunner*

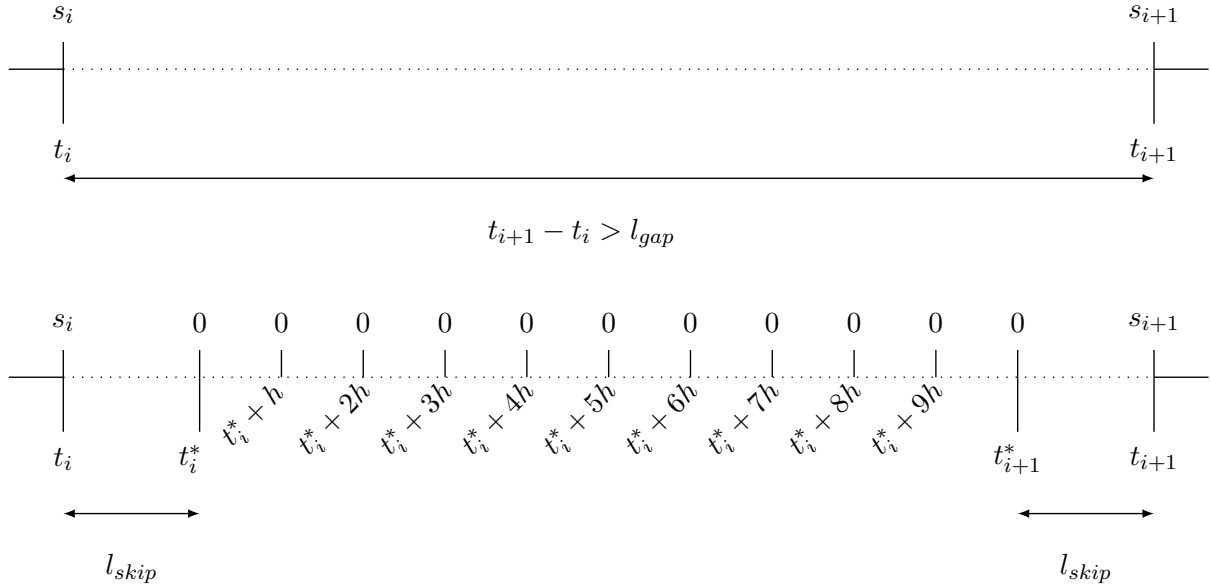


Figure 2: Illustration of the imputation process for speed with $m = 11$.

uses smart recording. It records key points where you change direction, speed, or heart rate” (Garmin Ltd. 2013). In both cases, interpolating directly to get the speed or power will lead to overestimation of the total workload within those intervals.

We assume that such intervals appear only when there is no significant work happening, and hence impute them with observations with zero speed (for running) or zero speed and power (for cycling).

Figure 2 shows a schematic representation of the imputation process for speed. The parameters l_{gap} , m and l_{skip} control the imputation, and can be specified via the `lgap`, `m` and `lskip` arguments of the `trackeRdata()` function, respectively.

If the observations at times t_i and t_{i+1} are more than l_{gap} seconds apart, then it is assumed that there is no significant work happening between t_i and t_{i+1} . The number of imputed records in the interval is m , and consists of two ‘outer’ records and $m - 2$ ‘inner’ records. The ‘outer’ records are l_{skip} seconds apart from the existing observations forming the beginning and the end of the interval, respectively. The ‘inner’ records are $h = (t_{i+1} - t_i - 2l_{skip}) / (m - 1)$ seconds apart.

The imputed records between t_i and t_{i+1} have zero speed or power, and the latitude, longitude and altitude measurements are set to their values at time t_i . All other variables are set to `NA`.

`trackeRdata()` also adds five records at the beginning and five at the end of a session, based on the assumption that there is no activity before and after the available records. These observations have zero speed or power, their latitude, longitude and altitude measurements are as in the first and last observations, respectively, and all other variables are set to `NA`. The imputed records are one second apart from each other and from the first and the last observation, respectively.

After the imputation process, the cumulative distances are updated based on the imputed

speeds and the time differences between consecutive observations, according to

$$d_{i+1} = d_i + s_i(t_{i+1} - t_i)$$

where s_i and d_i denote the speed and cumulative distance at time point t_i , respectively.

The following code chunk takes as input the raw data in `runDF` and constructs the corresponding “`trackeRdata`” object.

```
R> ## turn the raw data in runDF into a trackeRdata object
R> runTr0 <- trackeRdata(runDF)
```

The function `readContainer()` is a convenience wrapper that calls the suitable reading function and, then, `trackeRdata()` for the data processing and the organisation of the data in a “`trackeRdata`” object (see `?readContainer` for the available arguments). For example,

```
R> runTr1 <- readContainer(filepath, type = "tcx", timezone = "GMT")
R> identical(runTr0, runTr1)
```

```
[1] TRUE
```

The function `readDirectory()` allows the user to read all files of a supported format in a directory, rather than calling, e.g., `readContainer()` on each file separately. Using the argument `aggregate`, the user can decide if all data are first combined in a data frame and then split into sessions solely based on the time difference between consecutive observations. This way, e.g., warm-up and cool-down phases are put into the same session as the central part of training, even if they are recorded in separate container files. Alternatively, data from different container files are always stored in separate sessions.

trackeR ships with two “`trackeRdata`” objects containing 1 and 26 running sessions, respectively, and which can be loaded via

```
R> data(run, package = "trackeR")
R> data(runs, package = "trackeR")
```

We will use those objects for the illustrations throughout the paper.

5. Session summaries and visualisation

trackeR provides methods for summarising sessions in terms of scalar summaries, the time spent exercising in specified zones, the concept of work capacity, and distribution and concentration profiles.

5.1. Visualisation

For a first visual inspection of the data, the `plot()` method shows by default the evolution of heart rate and pace over the course of the selected sessions. For example, Figure 3 shows the evolution of heart rate and pace for the first three sessions in the `runs` object.

```
R> plot(runs, session = 1:3)
```

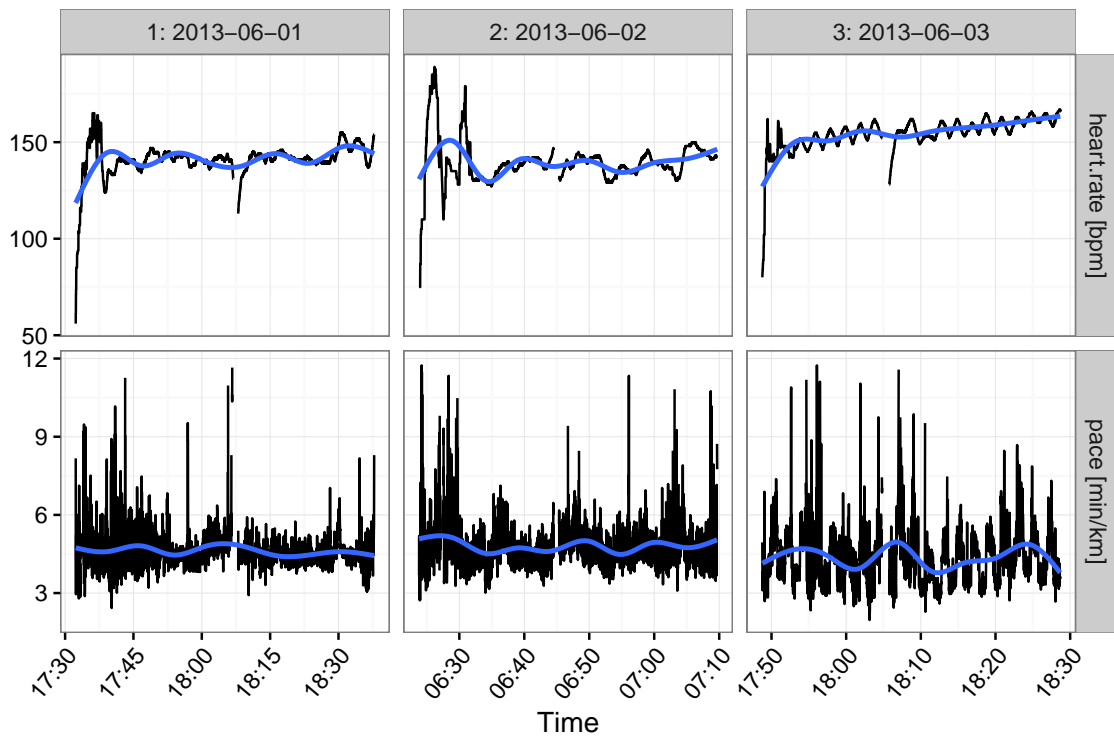



Figure 3: Heart rate and pace over the course of sessions 1–3.

The route covered during the session can also be displayed on a map via the `plotRoute()` method. The `plotRoute()` method uses the `ggmap` package (Kahle and Wickham 2013) and, hence, can work with the sources and maps supported by `ggmap`. For example, Figure 4 shows the route covered during session 4 in runs using a map downloaded from Open Street Map.

```
R> plotRoute(runs, session = 4, zoom = 13, source = "osm")
```

5.2. Scalar summaries

Each session can be summarised through common summary statistics using the `summary()` method. Such a session summary includes the total distance covered, the total duration, the time spent moving, work to rest ratio as well as averages of speed, pace, cadence, power, and heart rate, calculated based on total duration or the time spent moving. An athlete is considered to be moving if the speed is larger than some threshold s^* . This threshold can be set via the `movingThreshold` argument of the `summary()` method, and the package assumes that anything between not moving at all and walking with a speed below that threshold is resting. The default value for `movingThreshold` has been set to 1 meter per second, which is just below the speed humans prefer to walk at on average (1.4 meters per second; see Bohannon 1997).

The 'average speed moving' is calculated as total distance covered divided by time moving while 'average speed' is calculated as total distance divided by total duration. The average

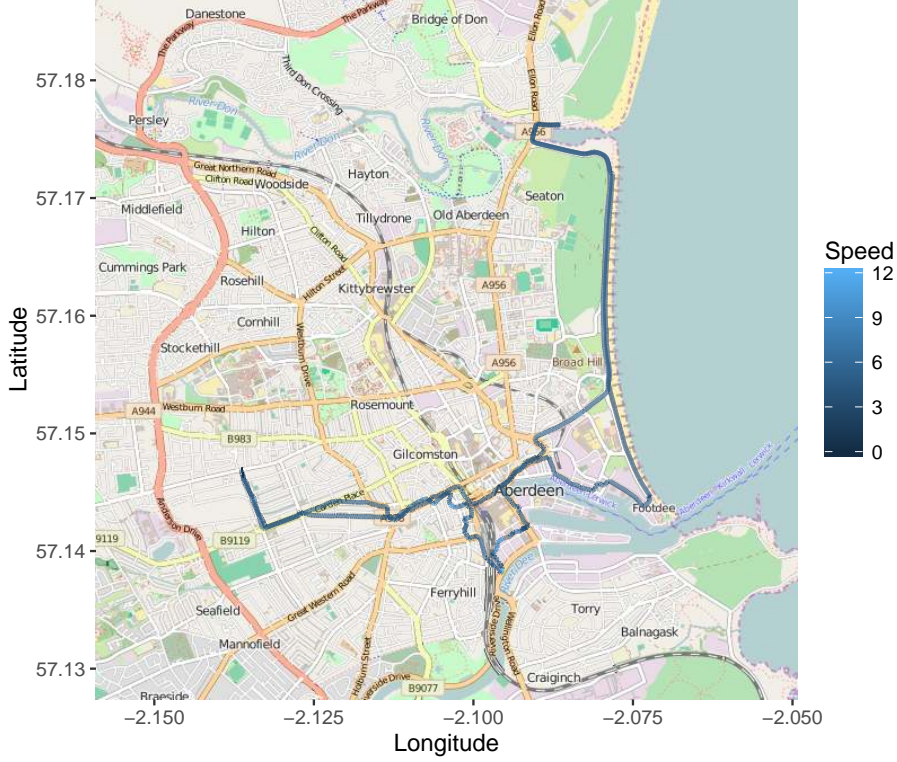


Figure 4: Route covered during session 4 on a map from Open Street Map.

pace (moving) is calculated as the inverse of the average speed (moving). The work to rest ratio is calculated as time moving divided by (total duration - time moving). The averages for cadence, power, and heart rate (total and moving) are weighted averages with weights depending on the time difference to the next observation. These averages also need to take into account missingness in the observations. For a variable of interest V , we can calculate a weighted mean for the total session while accounting for missing values via

$$\sum_i v_i \frac{\Delta_i K_i}{\sum_i \Delta_i K_i}$$

and its counterpart for the part of the session spent in motion via

$$\sum_i v_i \frac{\Delta_i K_i I(s_i > s^*)}{\sum_i \Delta_i K_i I(s_i > s^*)}$$

where v_i is the value of V at time point t_i , K_i is 1 if v_i is available, i.e., not missing, and 0 otherwise, $I(\cdot)$ denotes the indicator function, and $\Delta_i = t_i - t_{i-1}$ the time difference between observations at t_i and t_{i-1} .

The `summary()` method for “`trackeRdata`” objects returns an object of class “`trackeRdataSummary`” for which several methods are available. With the `print()` method, one can set the number of digits printed for the scalar summary statistics. The following example shows the summaries for sessions 1–2 with the default number of digits of 2 and then the summary of session 1 with 3 digits for comparison.

```
R> summary(runs, session = 1:2)
```

```
*** Session 1 ***
```

```
Session times: 2013-06-01 17:32:15 - 2013-06-01 18:37:56
Distance: 14130.7 m
Duration: 65.68 mins
Moving time: 64.17 mins
Average speed: 3.59 m_per_s
Average speed moving: 3.67 m_per_s
Average pace (per 1 km): 4:38 min:sec
Average pace moving (per 1 km): 4:32 min:sec
Average cadence: 88.66 steps_per_min
Average cadence moving: 88.87 steps_per_min
Average power: NA W
Average power moving: NA W
Average heart rate: 141.11 bpm
Average heart rate moving: 141.13 bpm
Average heart rate resting: 136.76 bpm
Work to rest ratio: 42.31
```

```
Moving threshold: 1 m_per_s
```

```
*** Session 2 ***
```

```
Session times: 2013-06-02 06:23:43 - 2013-06-02 07:09:47
Distance: 9450.24 m
Duration: 46.07 mins
Moving time: 44.13 mins
Average speed: 3.42 m_per_s
Average speed moving: 3.57 m_per_s
Average pace (per 1 km): 4:52 min:sec
Average pace moving (per 1 km): 4:40 min:sec
Average cadence: 88.21 steps_per_min
Average cadence moving: 88.25 steps_per_min
Average power: NA W
Average power moving: NA W
Average heart rate: 139.48 bpm
Average heart rate moving: 139.44 bpm
Average heart rate resting: 141.16 bpm
```

```
Work to rest ratio: 22.83
```

```
Moving threshold: 1 m_per_s
```

```
R> runSummary <- summary(runs, session = 1)
R> print(runSummary, digits = 3)
```

```
*** Session 1 ***
```

```
Session times: 2013-06-01 17:32:15 - 2013-06-01 18:37:56
Distance: 14130.7 m
Duration: 1.095 hours
Moving time: 1.069 hours
Average speed: 3.586 m_per_s
Average speed moving: 3.67 m_per_s
Average pace (per 1 km): 4:38 min:sec
Average pace moving (per 1 km): 4:32 min:sec
Average cadence: 88.664 steps_per_min
Average cadence moving: 88.874 steps_per_min
Average power: NA W
Average power moving: NA W
Average heart rate: 141.107 bpm
Average heart rate moving: 141.131 bpm
Average heart rate resting: 136.762 bpm
Work to rest ratio: 42.308
```

```
Moving threshold: 1 m_per_s
```

The `plot()` method shows the evolution of the various summary statistics over calendar time or over the course of the sessions. For example, the following code chunk produces Figure 5.

```
R> runSummaryFull <- summary(runs)
R> plot(runSummaryFull, group = c("total", "moving"),
+       what = c("avgSpeed", "distance", "duration", "avgHeartRate"))
```

5.3. Times in zones

A common way to summarise and characterise a session is to calculate how much time was spent exercising in certain zones, e.g., heart rate zones.

The `zones()` method for sessions returns an object of class “`trackeRdataZones`” for which methods `changeUnits()` and `plot()` are provided. The user can specify the variables, such as heart rate and speed, and their respective zones via the arguments `what` and `breaks`, respectively. Figure 6 shows a graphical representation of the zones summary, making it easier to see that more (relative) time was spent training with high speed ($> 4\text{ m/s}$) in sessions 3 and 4 than in sessions 1 and 2.

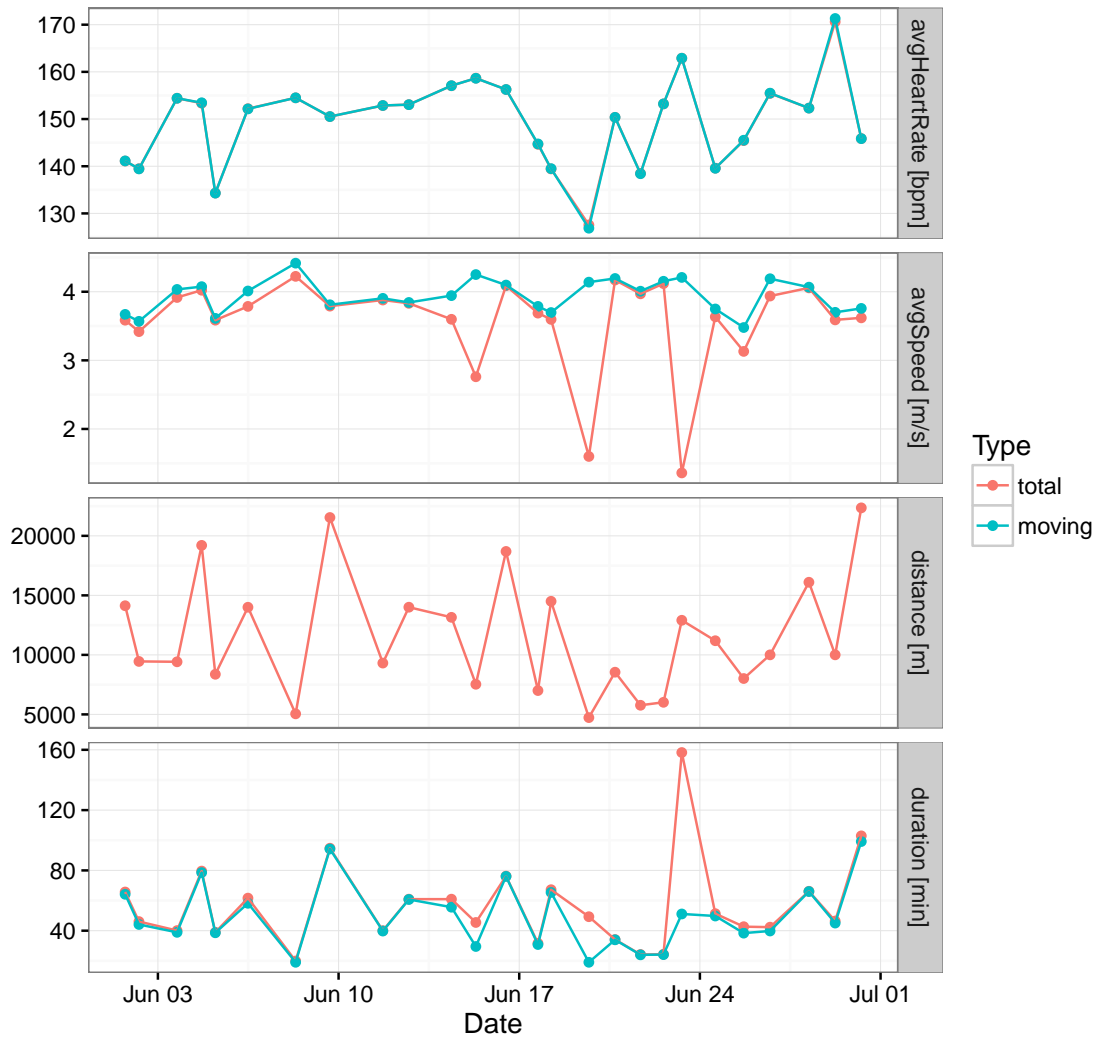


Figure 5: Selected session summaries for all 26 sessions.

```
R> runZones <- zones(runs[1:4], what = "speed", breaks = list(speed = c(0, 2:6, 12.5)))
R> ## if breaks is a named list, argument 'what' can be left unspecified
R> runZones <- zones(runs[1:4], breaks = list(speed = c(0, 2:6, 12.5)))
R> ## if only a single variable is to be evaluated, 'breaks' can also be a vector
R> runZones <- zones(runs[1:4], what = "speed", breaks = c(0, 2:6, 12.5))
R> plot(runZones)
```

5.4. Quantifying work capacity

The critical power model (Monod and Scherrer 1965) describes the relationship between the power output P and the time t_e to exhaustion at that power output

$$P = (W'_0/t_e) + CP \quad (1)$$

in terms of two parameters W'_0 and CP. The critical power (CP) is defined by Monod and

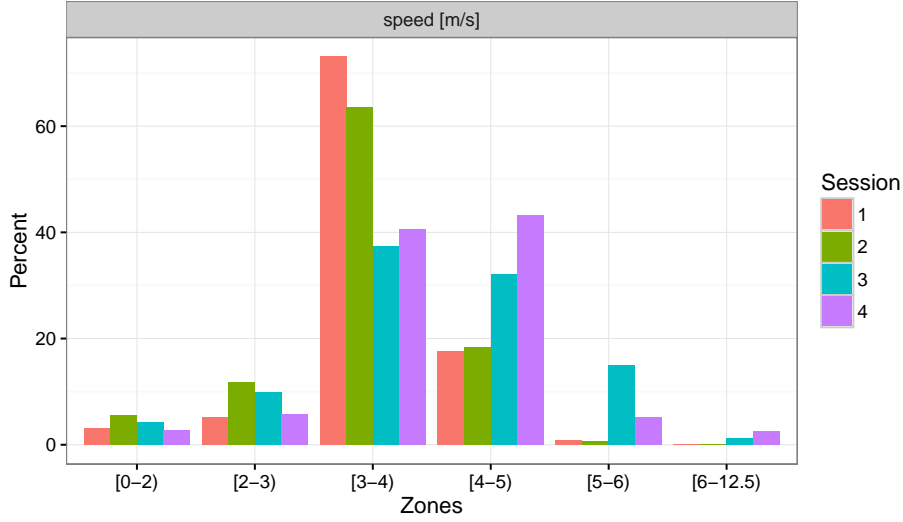


Figure 6: Zone summaries for speed of sessions 1–4.

Scherrer (1965) as “the maximum rate (of work) that [can be kept] up for a very long time without fatigue.” Skiba *et al.* (2012) describe CP as “a power output that could *theoretically* be maintained indefinitely on the basis of principally ‘aerobic’ metabolism.” W' (read W prime) represents a finite work capacity above CP. Skiba *et al.* (2012) assume that W' gets depleted during exercise with a power output above CP but also replenished during exercise with a power output of or below CP. We denote as W' the general concept of work capacity above CP, and $W'(t)$ is the state of W' at time t . The latter is also sometimes referred to as W' *balance* at time t . Additionally, the initial state of W' at the start of an exercise $t = t_0$ is $W'_0 = W'(t_0)$, which is one of the parameters in the critical power model (Equation 1). Total depletion of W'_0 results in the inability to produce a power output above CP. Thus, knowledge of the current state $W'(t)$, i.e., how much of that finite work capacity W'_0 is left at time t , is important to an athlete, particularly in a race.

While this concept is most commonly applied to cycling, where the power output is routinely measured, Skiba *et al.* (2012) suggest that it can also be applied to running, substituting power and critical power by speed and critical speed, respectively. For running, the model postulates that each runner has a finite capacity in terms of distance covered above the critical speed. Depending on how much the runner exceeds this critical speed, the finite capacity W'_0 is being exhausted in shorter times. Below we describe the models for depletion and replenishment of work capacity and how they are combined in **trackeR**.

Depletion of work capacity

Assuming constant power for periods of exertion above CP, Skiba, Fulford, Clarke, Vanhatalo, and Jones (2015) assume that W' is depleted at a rate directly proportional to the difference between the power output and CP

$$\frac{d}{dt}W'(t) = -(P - CP). \quad (2)$$

Solving Equation 2 for $W'(t)$ gives

$$W'(t) = -(P - CP)t + D \quad (3)$$

where $D \in \mathbb{R}$ is constant over t .

Suppose that the exercise over time $t_0 = 0$ to $t_n = T$ can be split into n intervals with breakpoints t_0, t_1, \dots, t_n such that the power output within each interval is constant, that is $P(t) = P_i$ for $t \in [t_{i-1}, t_i]$, $i \in \{1, \dots, n\}$. Then, using Equation 3, the change in $W'(t)$ over the interval can be expressed as

$$W'(t_i) - W'(t_{i-1}) = -(P_i - CP)(t_i - t_{i-1}). \quad (4)$$

Replenishing of work capacity

Skiba *et al.* (2015) assume that the periods with a power output at or below CP are periods of recovery during which W' is replenished with a rate that depends on the difference between CP and the power output, and the amount of W'_0 remaining, as follows:

$$\frac{d}{dt}W'(t) = \left(1 - \frac{W'(t)}{W'_0}\right)(CP - P). \quad (5)$$

Equation 5 assumes that recovery slows down as $W'(t)$ approaches the initial capacity W'_0 . Employing the substitution rule for integrals while solving Equation 5 and reexpressing in terms of $W'(t_{i-1})$ (see Appendix A for details) gives

$$W'(t_i) = W'_0 - (W'_0 - W'(t_{i-1})) \exp\left(\frac{P_i - CP}{W'_0}(t_i - t_{i-1})\right). \quad (6)$$

Since $W'(t_{i-1})$ is the amount of W'_0 remaining at the start of the interval $[t_{i-1}, t_i]$, $W'_0 - W'(t_{i-1})$ is the amount of W'_0 which has been depleted prior to t_{i-1} and not yet been replenished. Skiba *et al.* (2012) refer to this as W' expended. Skiba *et al.* (2015) describe the replenishing of W' indirectly by describing how W' expended is reduced over the course of such a recovery interval. The exponential decay factor used in Equation 6 here is the same as their Equation 4 with only different notation. Skiba *et al.* (2015) use t to describe the length of the interval, $D_{CP} = CP - P_i$ for the difference between critical power and power output, and W'_{exp} for the amount of W' previously expended. For $P_i < CP$, as is required for replenishment, $-D_{CP}$ and $P_i - CP$ are negative and thus the exponential factor is smaller than 1, leading to an exponential decay as described.

Skiba *et al.* (2012) also assume an exponential decay of previously expended W' to describe replenishing W' , albeit with a different decay factor. Instead of $(P_i - CP)/W'_0$, they use $1/\tau_{W'}$. The relationship between the time constant of replenishing $\tau_{W'}$ and the difference between critical power and recovery power \bar{P} is estimated based on experimental data as

$$\tau_{W'} = 546 \exp(-0.01(CP - \bar{P})) + 316$$

with recovery power \bar{P} estimated by the mean of all power outputs below CP.

Using Equation 6, i.e., the formulation of Skiba *et al.* (2015), the change in W' over the corresponding interval $[t_{i-1}, t_i]$ can be described through

$$W'(t_i) - W'(t_{i-1}) = (W'_0 - W'(t_{i-1})) \left(1 - \exp\left(\frac{P_i - CP}{W'_0}\Delta_i\right)\right). \quad (7)$$

Work capacity at time t_j

Equation 4 describes the depletion of W' (when $P_i > CP$) and Equation 7 describes replenishment of W' (when $P_i \leq CP$) over an interval $[t_{i-1}, t_i]$. These two aspects can be combined to describe the change over the interval as

$$W'(t_i) - W'(t_{i-1}) = - (P_i - CP)\Delta_i I(P_i > CP) + (W'_0 - W'(t_{i-1})) \left(1 - \exp\left(\frac{P_i - CP}{W'_0} \Delta_i\right) \right) (1 - I(P_i > CP)).$$

The amount of W' left at time point t_j , $j \in \{1, \dots, n\}$, can thus be described through the initial amount W'_0 and the changes happening in the j intervals of constant power previous to t_j

$$\begin{aligned} W'(t_j) &= W'_0 + \sum_{i=1}^j (W'(t_i) - W'(t_{i-1})) \\ &= W'_0 - \sum_{i=1}^j (P_i - CP)\Delta_i I(P_i > CP) + \\ &\quad \sum_{i=1}^j (W'_0 - W'(t_{i-1})) \left(1 - \exp\left(\frac{P_i - CP}{W'_0} \Delta_i\right) \right) (1 - I(P_i > CP)). \end{aligned} \quad (8)$$

W' expended at time t_j is then $W'_0 - W'(t_j)$.

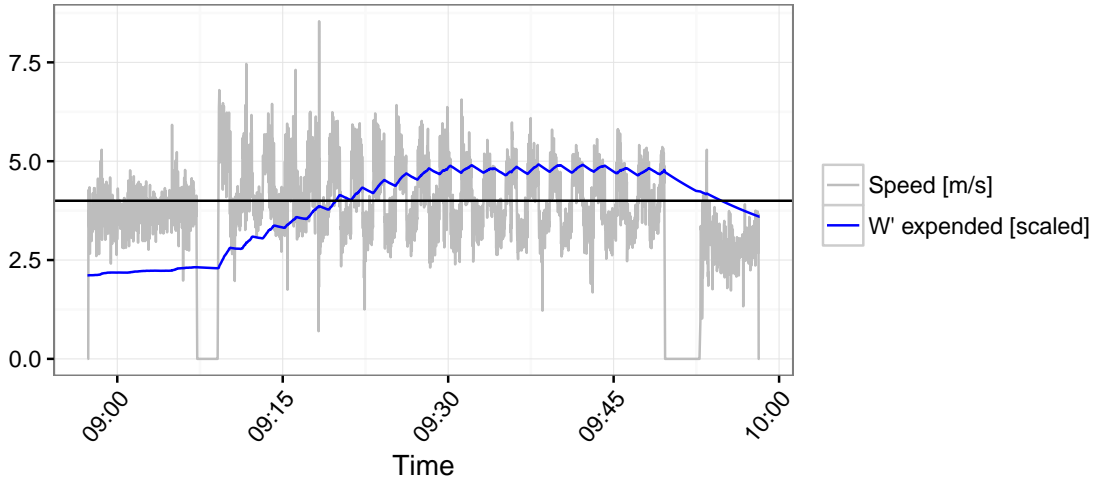
Function `Wprime()` can be used to calculate W' expended by setting argument `quantity` to "expended". If `quantity` is set to "balance", `Wprime()` calculates the current state $W'(t)$ (Equation 8). `Wprime()` contains implementations for Skiba *et al.* (2012) and Skiba *et al.* (2015), which can be selected via the `version` argument. For example, session 11 of the example data is an interval training with a warm-up and cool-down phase. Assuming a critical speed of 4 meters per second, the following code chunk produces Figure 7, which shows W' expended, based on the specification of Skiba *et al.* (2012), along with the corresponding speed profile.

```
R> wexp <- Wprime(runs, session = 11, quantity = "expended",
+               cp = 4, version = "2012")
R> plot(wexp, scaled = TRUE)
```

During the warm-up phase speed rarely exceeds 4 meters per second and W' expended remains low. Over the course of the interval training, W' expended rises during the high-intensity phases and drops during the recovery phases. In the last part of the session, speeds are mostly below 4 meters per second and W' expended drops again.

5.5. Distribution and concentration profiles

Kosmidis and Passfield (2015) introduce the concept of distribution profiles for which the **trackeR** package provides an implementation. These profiles are motivated by the need to compare sessions and use information on such variables as heart rate or speed during a session for further modelling.

Figure 7: W' expended in session 11.

For a session lasting t_n seconds, the distribution profile is defined as the curve $\{v, \Pi(v) | v \geq 0\}$ where

$$\Pi(v) = \int_0^{t_n} I(v(t) > v) dt.$$

The function $\Pi(v)$ is monotone decreasing and describes the time spent exercising above a threshold v for a variable V under consideration (e.g., heart rate or speed).

On the basis of observations v_0, \dots, v_n for V , at respective time points t_0, \dots, t_n , the observed version of $\Pi(v)$ can be calculated as

$$P(v) = \sum_{i=1}^n (t_i - t_{i-1}) I(v_i > v).$$

This can subsequently be smoothed respecting the positivity and monotonicity of $\Pi(v)$, e.g., via a shape constraint additive model with Poisson responses (Pya and Wood 2015).

The concentration profile is defined in Kosmidis and Passfield (2015) as the negative derivative of a distribution profile and is suitable for revealing concentrations of time around certain values of the variable under consideration.

Distribution profiles can be calculated using the `distributionProfile()` function which returns an object of class “`distrProfile`”. Concentration profiles can be derived from distribution profiles using `concentrationProfile()`, which returns an object of class “`conProfile`”. Table 1 includes an overview of constructor functions and available methods for distribution and concentration profiles.

By default, distribution profiles are calculated for speed and heart rate on grids covering the ranges of $[0, 12.5]$ meters per second and $[0, 250]$ beats per minute, respectively. The following code chunk illustrates the use of `distributionProfile()` and shows how users can specify the variables for which to calculate profiles and the respective grids.

```
R> dProfile <- distributionProfile(runs, session = 1:4,
+   what = c("speed", "heart.rate"),
```

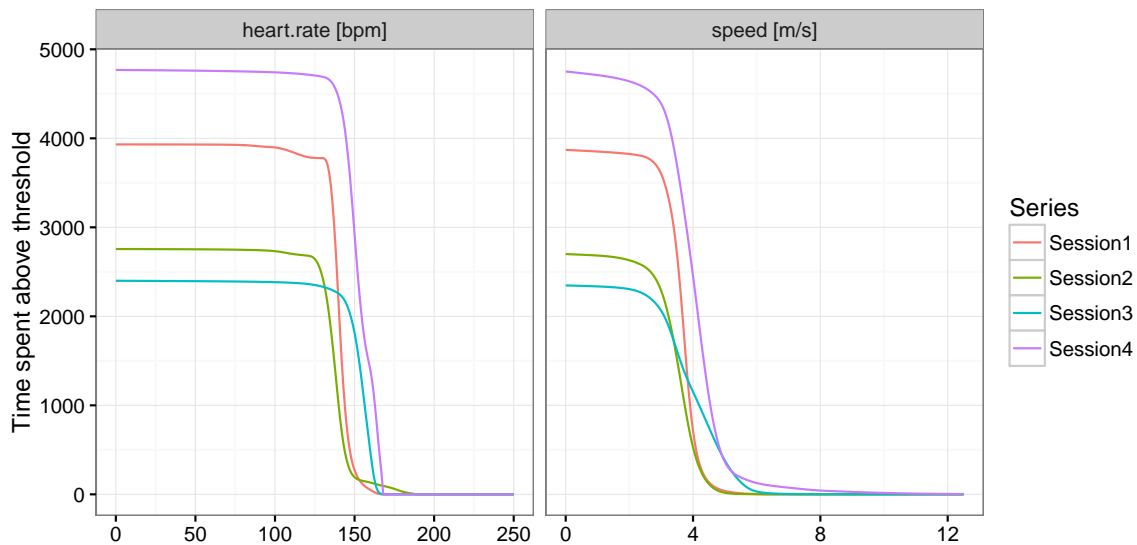


Figure 8: Distribution profiles for sessions 1–4.

```
+ grid = list(speed = seq(0, 12.5, by = 0.05), heart.rate = seq(0, 250)))
R> plot(dProfile, multiple = TRUE)
```

The `multiple` argument of the `plot()` method determines whether to plot the profiles in separate panels (`FALSE`) or overlay them in a common panel (`TRUE`), as in Figure 8. The different session lengths are clearly visible in the height of the curves at 0. Amongst the distribution profiles for speed, the descent of the profile for session 3 is slower than for the other sessions. This difference is most apparent in the concentration profiles, which are shown in Figure 9 and are produced by the following code chunk.

```
R> cProfile <- concentrationProfile(dProfile, what = "speed")
R> plot(cProfile, multiple = TRUE)
```

The profile for session 3 has a mode at around 3.5 meters per second and another one at 5 meters per second, showing that this session involved training at a combination of low and high speeds.

6. Handling units of measurement

Data objects of class “`trackeRdata`” and all objects derived from these (“`trackeRdataSummary`”, “`trackeRdataZones`”, “`trackeRWprime`”, “`distrProfile`”, and “`conProfile`”) carry an attribute with the relevant units of measurement. The `getUnits()` method returns the units of measurement for each variable and the `changeUnits()` method can be used to change one or more variables from one set of units to another. For example,

```
R> ## get the units for the variables in run
R> getUnits(run)
```

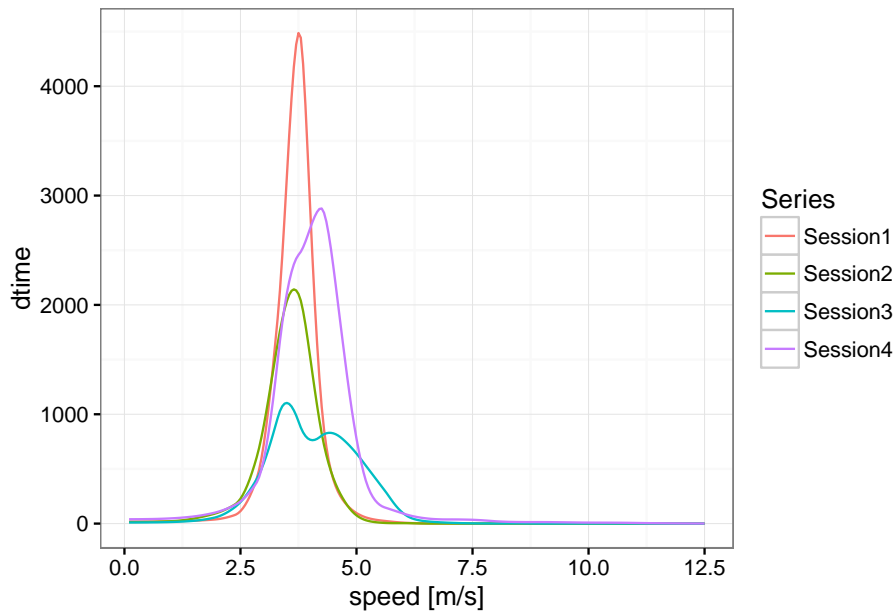


Figure 9: Concentration profiles for sessions 1–4.

	variable	unit
1	latitude	degree
2	longitude	degree
3	altitude	m
4	distance	m
5	heart.rate	bpm
6	speed	m_per_s
7	cadence	steps_per_min
8	power	W
9	pace	min_per_km
10	duration	s

```
R> ## change the unit of speed into miles per hour
R> runTr2 <- changeUnits(run, variable = "speed", unit = "mi_per_h")
R> getUnits(runTr2)
```

	variable	unit
1	latitude	degree
2	longitude	degree
3	altitude	m
4	distance	m
5	heart.rate	bpm
6	speed	mi_per_h
7	cadence	steps_per_min
8	power	W
9	pace	min_per_km
10	duration	s

Measurement	Unit(s)
latitude	degrees (degree , default)
longitude	degrees (degree , default)
altitude	meters (m , default), kilometres (km), miles (mi), feet (ft)
distance	meters (m , default), kilometres (km), miles (mi), feet (ft)
speed	meters per second (m_per_s , default), kilometres per hour (km_per_h), feet per minute (ft_per_min), feet per second (ft_per_s), miles per hour (mi_per_h)
cadence	steps per minute (steps_per_min , default for running), revolutions per minute (rev_per_min , default for cycling)
power	Watts (W , default), kilowatts (kW)
heart rate	beats per minute (bpm , default)
pace	minutes per kilometre (min_per_km , default), minutes per mile (min_per_mi), seconds per meter (s_per_m)
duration	seconds (s), minutes (min), hours (h) – default is the largest possible unit for which the duration is larger than 1

Table 2: Supported units of measurement.

Table 2 shows the variables and the corresponding units that are currently supported in **trackeR**.

If objects with different units are `c()`ombined in one object, then the units of the first session are applied to all other sessions. Furthermore, the `changeUnits()` method uses name matching to figure out which conversion needs to be done. This allows the user to easily add support for converting from `unitOld` to `unitNew` by authoring a function named `unitOld2unitNew`.

If we wish to report the speed summaries for session 1 in `runSummary` in feet per hour (not currently supported) instead of meters per second, we need to simply provide the appropriately named conversion function as illustrated below. Note that the conversion applies to all speed summaries, i.e., to 'average speed' and 'average speed moving'.

```
R> m_per_s2ft_per_h <- function(x) x * 3937/1200 * 3600
R> changeUnits(runSummary, variable = "speed", unit = "ft_per_h")
```

```
*** Session 1 ***
```

```
Session times: 2013-06-01 17:32:15 - 2013-06-01 18:37:56
```

```
Distance: 14130.7 m
```

```
Duration: 1.09 hours
```

```
Moving time: 1.07 hours
```

```
Average speed: 42349.08 ft_per_h
```

```
Average speed moving: 43350.06 ft_per_h
```

```
Average pace (per 1 km): 4:38 min:sec
```

```
Average pace moving (per 1 km): 4:32 min:sec
```

```
Average cadence: 88.66 steps_per_min
```

```
Average cadence moving: 88.87 steps_per_min
```

```
Average power: NA W
```

Variable	Unit	Lower threshold	Upper threshold
latitude	degrees	-90	90
longitude	degrees	-180	180
altitude	meter	-500	9000
distance	meter	0	∞
heart rate	beats per minute	0	250
speed	meters per second	0	12.5 (100)
cadence	steps (revolutions) per minute	0	∞
power	Watts	0	∞
pace	minutes per kilometre	0	∞
duration	seconds	0	∞

Table 3: Default thresholds for running data, values in parentheses apply to cycling data.

```
Average power moving: NA W
Average heart rate: 141.11 bpm
Average heart rate moving: 141.13 bpm
Average heart rate resting: 136.76 bpm
Work to rest ratio: 42.31
```

```
Moving threshold: 11811 ft_per_h
```

7. Thresholding and smoothing

There are instances where the data include artefacts due to inaccuracies in the GPS measurements. These are handled with the `threshold()` method for objects of class “`trackeRdata`”, which replaces values outside the specified thresholds with `NA`. The variables and the (lower and upper) thresholds which should be applied for each variable can be specified through the arguments `variable`, `lower`, and `upper`, respectively. An example is given in `?threshold`. The default thresholds are listed in Table 3 and, if necessary, are converted to the units of measurement used for the “`trackeRdata`” object.

The other option for data handling is the `smoother()` method for “`trackeRdata`” objects. This applies a summarising function, such as the mean or median, over a rolling window. Both operations `threshold()` and `smoother()` are used in the `plot()` method for “`trackeRdata`” objects. The default settings for `plot()` are to apply the thresholds specified in Table 3 but not to smooth the data. The top left panel in Figure 10 gives an example where no thresholds are applied and the top right panel uses default settings. The spike to over 20 meters per second in the top left panel is clearly an error in the data; the current world record for 100 meters (by Usain Bolt, August 16, 2009) is 9.58 seconds which translates to an average speed of 10.44 meters per second. The bottom panels show the effect of first applying the default thresholds and then smoothing the data through a rolling median with a window width of 20 observations, either done within the `plot()` method (bottom left) or explicitly via the `threshold()` and `smoother()` methods (bottom right).

```
R> ## without thresholds
R> plot(runs, session = 4, what = "speed", threshold = FALSE)
```

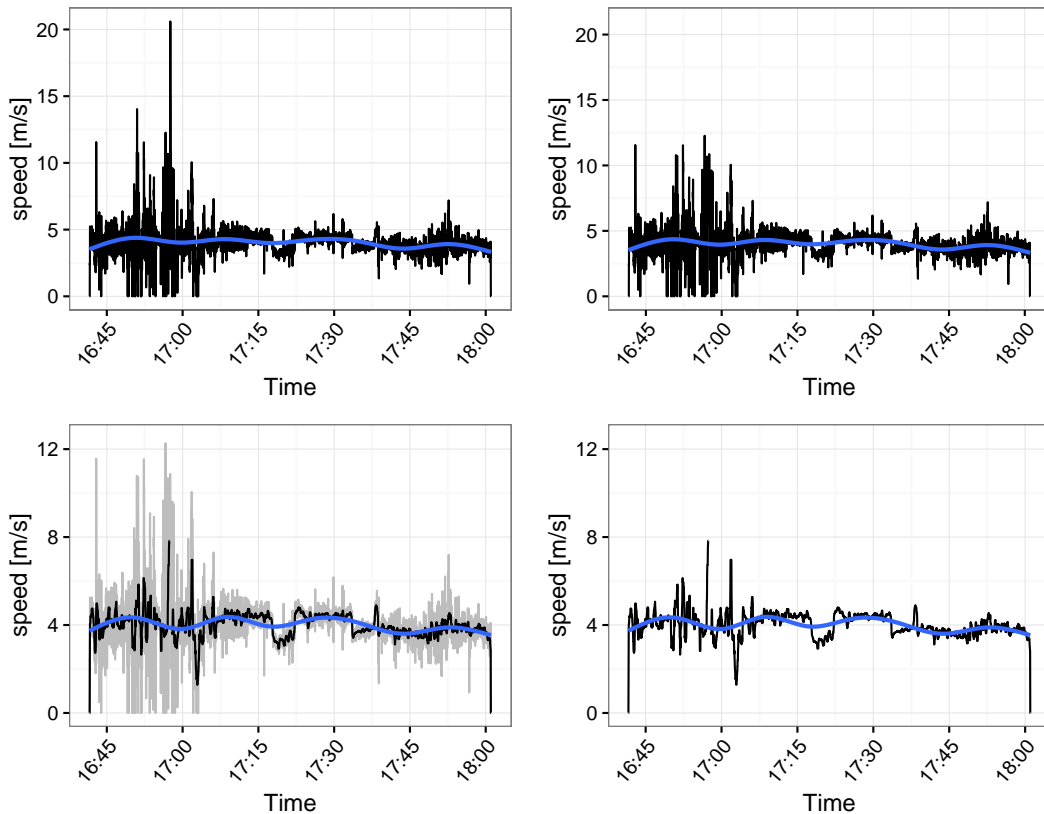


Figure 10: Speed profile of session 4 without thresholding (top left), with the default settings (top right), and with default thresholds as well as smoothing through a rolling median over a window of 20 observations done within the plot function (bottom left) and separately (bottom right).

```
R> ## with default thresholds
R> plot(runs, session = 4, what = "speed")
R> ## with default thresholds and smoothing
R> plot(runs, session = 4, what = "speed", smooth = TRUE, fun = "median", width = 20)
R> ## thresholding and smoothing outside of plot method
R> run4 <- threshold(runs[4])
R> run4S <- smoother(run4, what = "speed", fun = "median", width = 20)
R> plot(run4S, what = "speed", smooth = FALSE)
```

The method `smoother()` is also available for distribution and concentration profiles. Smoothing a distribution profile requires a smoothing technique which respects the positivity and monotonicity of the distribution profile. This can be achieved by fitting a shape constrained additive model with Poisson responses as implemented in the `scam` package (Pya 2015). When smoothing concentration profiles, the raw profiles are transformed to distribution profiles which are subsequently smoothed preserving the positivity and monotonicity. The smooth concentration profiles are then derived from the smoothed distribution profiles. The `plot()` methods for “`distrProfile`” and “`conProfile`” smooth the profiles prior to plotting by default.

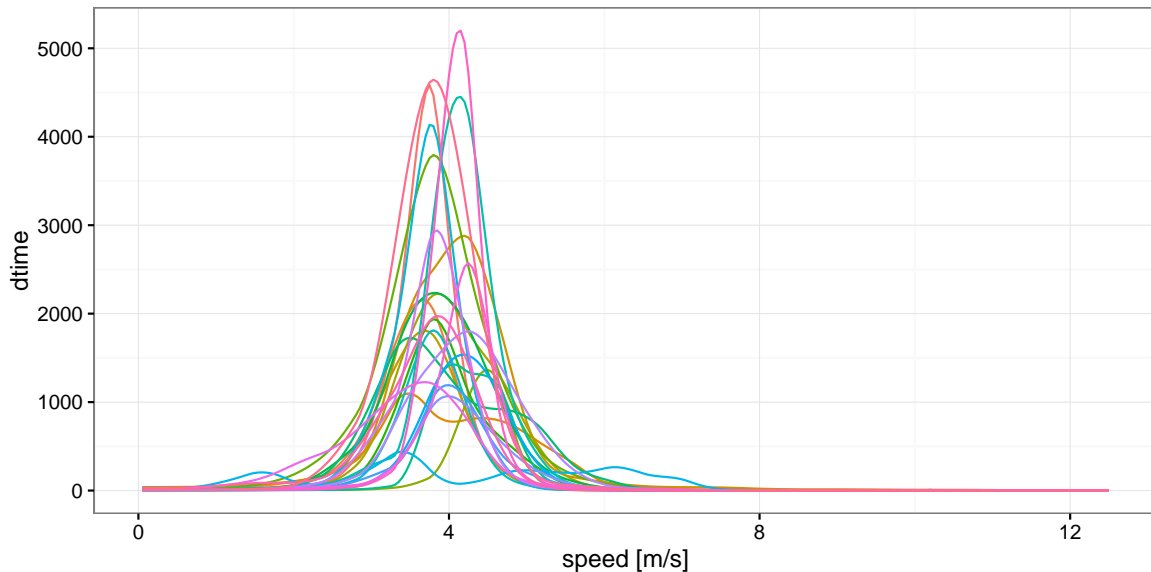


Figure 11: Smoothed speed concentration profiles for all 26 sessions.

8. Case study

The example data set included in the package contains 26 sessions of a single male runner in June 2013. A visualisation of scalar summaries for the sessions can be found in Figure 5. The distance covered in those sessions ranges from 4.73 km to 22.35 km, and most sessions were spent moving almost the entire time.

The code chunk below calculates the smoothed distribution profiles for the 26 sessions. The corresponding concentration profiles are shown in Figure 11.

```
R> library("tracker")
R> ## load data
R> data(runs, package = "tracker")
R> ## apply default thresholds
R> runsT <- threshold(runs)
R> ## get and smooth distribution profiles
R> dpRuns <- distributionProfile(runsT, what = "speed")
R> dpRunsS <- smoother(dpRuns, cores = 2)
R> ## get concentration profiles
R> cpRuns <- concentrationProfile(dpRunsS)
```

The majority of the profiles for speed concentrate around 4 meters per second. However, the curves vary in their shape (unimodal or multimodal), height, and location (revealing concentrations at higher or lower speeds). In an attempt to characterise the differences between curves, we use a functional principal components analysis (PCA, e.g., Ramsay and Silverman 2005). Here, we use the implementation provided in the `pca.df()` function of the `fda` R package (Ramsay *et al.* 2014). After preparing the concentration profiles in the required functional data format, a functional PCA with four principal components is fitted.

```

R> ## prepare functional data
R> library("fda")
R> gridSpeed <- seq(0, 12.5, length = 251)
R> sp <- matrix(unlist(cpRuns$speed), ncol = 250, byrow = TRUE,
+             dimnames = list(names(cpRuns$speed), gridSpeed[-1]))
R> spfd <- Data2fd(argvals = gridSpeed[-1], y = t(sp))
R> ## fit functional PCA
R> sppca <- pca.fd(spfd, nharm = 4)
R> ## share of variance
R> varprop <- round(sppca$varprop * 100); names(varprop) <- 1:4
R> varprop

```

```

  1  2  3  4
64 27  6  2

```

The first two harmonics capture 91% of the variation between curves. Since further harmonics capture considerably less variation, only the first two are chosen for further inspection.

Figure 12 shows the mean function (solid line) and the variation captured in the two harmonics (between the dashed and dotted lines). The first harmonic (top panel) shows that the most important characteristic of the concentration profiles is the relative value of the profiles, which is closely related to the overall session duration. The left panel of Figure 13 shows the score on the first harmonic versus 'duration moving' which is calculated as part of the scalar session summaries. The second harmonic (bottom panel of Figure 12) shows variation along the speed thresholds in the centre of the curve. This variation can be explained well by the scalar measure 'average speed moving' as shown in the right panel of Figure 13.

The concentration profiles and a functional PCA thus indicate that the two scalar summaries 'duration moving' and 'average speed moving' provide a good summary of the speed information in the sessions and can be used, for example, in order to incorporate speed as covariance information in further regression analyses.

9. Concluding remarks

Package **trackeR** provides tools to import data from GPS-enabled tracking devices in R. Currently it supports TCX and db3 formats, but the package structure allows the easy inclusion of other formats. After careful processing, the data are stored in a session-based, unit-aware, and operation-aware objects. Core infrastructure for handling units of measurement as well as for summarising and visualising tracking data is provided. Time in zones, work capacity, and distribution and concentration profiles can be derived and used in further statistical analyses, such as the functional principal component analysis of speed concentration profiles presented here.

Areas for further development include the extension of the reading capabilities to more file formats and the implementation of more analytic tools, such as record power profiles (Pinot and Grappe 2011). The case study in Section 8 draws some links to functional data analysis. We are also investigating options for including capabilities for the analysis of the positional information.

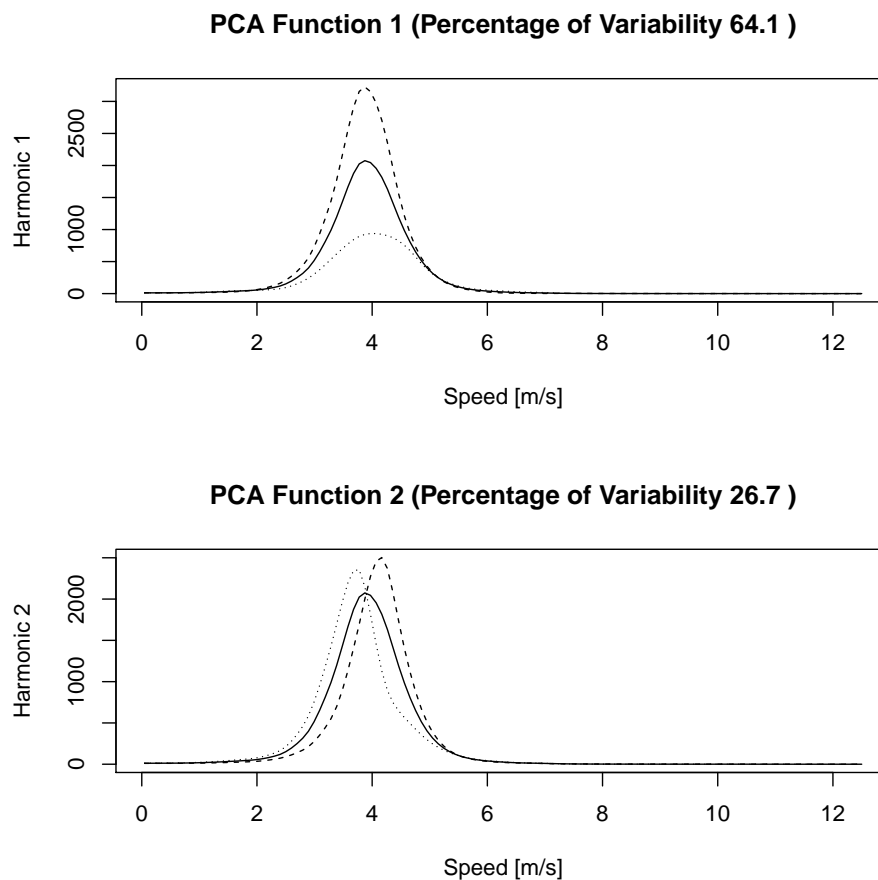


Figure 12: Harmonics 1–2 for the speed concentration profiles. Mean function (solid line) with suitable multiples of the harmonic added (dashed line) and subtracted (dotted line).

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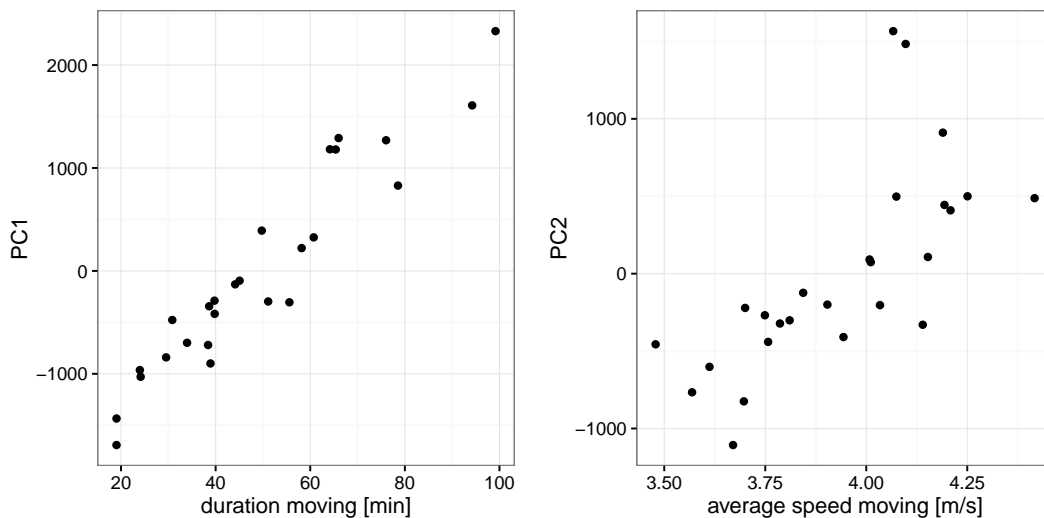


Figure 13: PC1 score vs. 'duration moving' (left) and PC2 score vs. 'average speed moving' (right).

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A. Replenishment of W'

Assuming that power is constant, the solution of the differential equation describing the rate of replenishment in Equation 5 with respect to $W'(t)$ gives

$$1 - \frac{W'(t)}{W'_0} = \exp\left(\frac{P - CP}{W'_0}t + D/W'_0\right). \quad (9)$$

Using Equation 9 over an interval $[t_{i-1}, t_i)$ of constant power gives

$$1 - \frac{W'(t_i)}{W'_0} = \exp\left(\frac{P_i - CP}{W'_0}(t_i - t_{i-1})\right) \left(1 - \frac{W'(t_{i-1})}{W'_0}\right).$$

Hence, $W'(t_i)$ can be expressed in terms of $W'(t_{i-1})$ as

$$W'(t_i) = W'_0 - (W'_0 - W'(t_{i-1})) \exp\left(\frac{P_i - CP}{W'_0}(t_i - t_{i-1})\right).$$

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